1 Meixner-Pollaczeck to Laguerre

1. *Asymptotic expansion for large λ:*

$$P_n^{(\lambda)}(x;\phi) = \sum_{k=0}^{n} c_k L_{n-k}^{(\alpha)}(X), \qquad X = \alpha + 1 - 2\lambda\cos\phi - 2x\sin\phi.$$

2. *Coefficients:*

$$c_0 = 1, \quad c_1 = 0, \quad c_2 = x\sin(2\phi) + \lambda\cos(2\phi) - 2(x\sin\phi + \lambda\cos\phi) + \frac{\alpha}{2}$$

and, for $k = 4, 5, 6, \ldots$,

$$\begin{aligned} kc_k = \; & 2(1+\cos\phi)(k-1)c_{k-1} + [\alpha + 1 - 2\lambda + 4(\cos\phi - 1)(\lambda\cos\phi + x\sin\phi) + \\ & 2(2-k)(1+2\cos\phi)]c_{k-2} + [4\lambda + 2(k-3)(1+\cos\phi) - \\ & 2(\alpha+1)\cos\phi]c_{k-3} + (\alpha + 5 - k - 2\lambda)c_{k-4}. \end{aligned}$$

3. *Asymptotic property:*

$$c_k L_{n-k}^{(\alpha)}(X) = \mathcal{O}(\lambda^{n+[k/2]-k}), \qquad \lambda \to \infty.$$

4. *Limit (Known):*

$$\lim_{\phi\to 0} P_n^{(\alpha+1)/2}\left(-\frac{x}{2\phi};\phi\right) = L_n^{(\alpha)}(x).$$

2 Jacobi to Laguerre

1. *Asymptotic expansion for large $\alpha + \beta$:*

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} c_k L_{n-k}^{(\alpha)}(X), \qquad X = \frac{1}{2}(\alpha + \beta + 2)(1-x).$$

2. *Coefficients:*

$$c_0 = 1, \quad c_1 = 0, \quad c_2 = \frac{1}{8}[3\beta - \alpha - 2(\alpha + 3\beta + 4)x + (3\alpha + 3\beta + 8)x^2].$$

3. *Asymptotic property:*

$$c_k\, L_{n-k}^{(\alpha)}(X) = \mathcal{O}((\alpha+\beta)^{n+[k/2]-k}), \qquad \alpha + \beta \to \infty, \quad \frac{\alpha - \beta}{\alpha + \beta} \to 0.$$

4. *Limit (known):*

$$\lim_{\beta\to\infty} P_n^{(\alpha,\beta)}\left(1 - \frac{2x}{\beta}\right) = L_n^{(\alpha)}(x).$$

3 Meixner to Laguerre

1. *Asymptotic expansion for large β:*

$$M_n(x;\beta,c)=\sum_{k=0}^{n} c_k L_{n-k}^{(\alpha)}(X),\qquad X=\alpha-\beta+1+\frac{1-c}{c}x.$$

2. *Coefficients:*

$$c_0=1,\quad c_1=0,\quad c_2=\frac{1+\alpha-\beta}{2}+\frac{2c-c^2-1}{2c^2}x.$$

3. *Asymptotic property:*

$$c_k L_{n-k}^{(\alpha)}(X)=\mathcal{O}(\beta^{n+[k/2]-k}),\qquad \beta\to\infty.$$

4. *Limit (known):*

$$\lim_{c\to 1} M_n\left(\frac{cx}{1-c};\alpha+1,c\right)=\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)}.$$

4 Krawtchouk to Laguerre

1. *Asymptotic expansion for large N:*

$$\binom{N}{n} K_n(x;p,N)=\sum_{k=0}^{n} c_k L_{n-k}^{(\alpha)}(X),\qquad X=\alpha+1-N+\frac{x}{p}.$$

2. *Coefficients:*

$$c_0=1,\quad c_1=0,\quad c_2=\frac{1}{2}\left[1+\alpha-3N+\frac{1}{p}\left(4-\frac{1}{p}\right)x\right].$$

3. *Asymptotic property:*

$$c_k\, L_{n-k}^{(\alpha)}(X)=\mathcal{O}(N^{n+[k/2]-k}),\qquad N\to\infty.$$

5 Continuous Dual Hahn to Laguerre

1. *Asymptotic expansion for large a and b:*

$$\frac{S_n^{a,b,c}(x^2)}{(a+b)_n n!}=\sum_{k=0}^{n} c_k L_{n-k}^{X}(A),$$

with

$$A=p_1(x)+p_1^2(x)-2p_2(x),\qquad\qquad X=A+p_1(x)-1,$$

$$\begin{aligned}
p_1(x) &= c+\frac{ab-x^2}{a+b},\\
p_2(x) &= \frac{c(c+1)}{2}+\frac{abc}{a+b}+\frac{ab(1+a)(1+b)-[c+2ab+(1+c)(1+2(a+b))]x^2+x^4}{2(a+b)(1+a+b)},
\end{aligned}$$

and

$$f(w)=e^{Aw/(1-w)}(1-w)^{X+1}(1-w)^{-c+ix}{}_2F_1\left(\left.\begin{array}{c} a+ix,\,b+ix \\ a+b\end{array}\right| w\right).$$

2. *Asymptotic property:*

$$c_k L_{n-k}^{X}(A)=\mathcal{O}\left(a^{n+[\frac{k}{3}]-k}\right)\quad\text{when}\quad a,\,b\to\infty\quad\text{with}\quad a\sim b.$$

when $a,b\to\infty$ uniformly in c with $a\sim b$ and c/a bounded.

6 Continuous Hahn to Laguerre

1. *Asymptotic expansion for large a and b:*

$$\frac{(2a+2b-1)_n P_n^{\alpha,\beta,\overline{\alpha},\overline{\beta}}(x)}{(2a)_n(a+b+i(c-d))_n i^n} = \sum_{k=0}^{n} c_k L_{n-k}^{X}(A),$$

$$A = p_1(x) + p_1^2(x) - 2p_2(x), \qquad X = A + p_1(x) - 1,$$

$$p_1(x) = \; i\,(1-2a-2b)\,\frac{bc+ad+x(a+b)}{a[a+b+i(c-d)]},$$

$$p_2(x) = \; (2a+2b-1)(a+b)\Big\{1 - \frac{(2a+2b+1)[a+i(c+x)]}{a[a+b+i(c-d)]}$$

$$+\frac{(2a+2b+1)(2a+2b+2)[a+i(c+x)][1+a+i(c+x)]}{2a(2a+1)[a+b+i(c-d))][a+b+1+i(c-d)]}\Big\},$$

$$f(w) = e^{Aw/(1-w)}(1-w)^{X+2(1-a-b)}\,{}_3F_2\left(\begin{array}{c} a+b-\frac{1}{2}, a+b, a+i(c+x) \\ 2a, a+c+i(b-d) \end{array}\middle| -\frac{4w}{(1-w)^2}\right).$$

2. *Asymptotic property:*

$$c_k L_{n-k}^{X}(A) = \mathcal{O}\left(b^{[\frac{n-k}{2}]+[\frac{k}{3}]}\right) \quad \text{when} \quad a,\, b \to \infty \quad \text{with} \quad a \sim b.$$

7 Hahn to Laguerre

1. *Asymptotic expansion for large a, b and N:*

$$\frac{(a+b+1)_n Q_n^{a,b,N}(x)}{n!} = \sum_{k=0}^{n} c_k L_{n-k}^{X}(A),$$

with A and X

$$A = p_1(x) + p_1^2(x) - 2p_2(x), \qquad X = A + p_1(x) - 1,$$

and

$$p_2(x) \;= (a+b+1)(a+b+2)\left[\frac{1}{2} - \frac{x(a+b+3)}{N(a+1)} + \frac{x(x-1)(a+b+3)(a+b+4)}{2N(N-1)(a+1)(a+2)}\right],$$

$$p_1(x) \;= (a+b+1)\left[1 - \frac{x(a+b+2)}{N(a+1)}\right],$$

$$f(w) = e^{Aw/(1-w)}(1-w)^{X-a-b}\,{}_3F_2\left(\begin{array}{c} (a+b+1)/2, (a+b+2)/2, -x \\ a+1, -N \end{array}\middle| -\frac{4w}{(1-w)^2}\right).$$

2. *Asymptotic property:*

$$c_k L_{n-k}^{X}(A) = \mathcal{O}\left(N^{n-k}\right) \quad \text{when} \quad a, b, N \to \infty \quad \text{with} \quad a \sim b \sim N.$$

8 Dual Hahn to Laguerre

1. *Asymptotic expansion for large a and N:*

$$\frac{(-N)_n R_n^{a,b,N}(\lambda(x))}{n!} = \sum_{k=0}^{n} c_k L_{n-k}^{X}(A), \quad \lambda(x) \equiv x(x+a+b+1),$$

$$A = p_1(x) + p_1^2(x) - 2p_2(x), \qquad X = A + p_1(x) - 1,$$

$$\begin{aligned} p_1(x) &= \frac{\lambda(x)}{a+1} - N, \\ p_2(x) &= \frac{(N-x)(N-x-1)}{2} + \frac{x(b+x)\left[2(2+a)(x-N) + (b+x-1)(x-1)\right]}{2(a+1)(2+a)}, \end{aligned}$$

$$f(w) = e^{Aw/(1-w)}(1-w)^{X+1}(1-w)^{N-x}{}_2F_1\left(\left.\begin{array}{c} -x, -b-x \\ a+1 \end{array}\right| w\right).$$

2. *Asymptotic property:*

$$c_k L_{n-k}^{X}(A) = \mathcal{O}\left(N^{n-k-1}\right) \quad \text{when} \quad N,\, a \to \infty \quad \text{with} \quad N \sim a.$$