1 Laguerre to Hermite

1. *Asymptotic expansion for large α:*

$$L_n^{\alpha}(x) = (-1)^n\, B^n \sum_{k=0}^{n} \frac{c_k}{B^k}\, \frac{H_{n-k}(X)}{(n-k)!},$$

$$B = \sqrt{x - \frac{\alpha+1}{2}}, \quad X = \frac{x-\alpha-1}{2B}.$$

2. *Coefficients:*

$$c_0 = 1, \quad c_1 = c_2 = 0, \quad c_3 = \frac{1}{3}(3x-\alpha-1)$$

and, for $k = 4, 5, 6, \ldots$,

$$kc_k = -2(k-1)c_{k-1} - (k-2)c_{k-2} + (3x-\alpha-1)c_{k-3} + (2x-\alpha-1)c_{k-4}.$$

3. *Asymptotic property:*

$$\frac{c_k}{B^k}\, H_{n-k}(X) = \mathcal{O}(\alpha^{n+\lfloor k/3\rfloor - 3k/2}), \qquad \alpha \to \infty.$$

4. *Limit (known):*

$$\lim_{\alpha\to\infty} \alpha^{-n/2} L_n^{\alpha}(x\sqrt{\alpha}+\alpha) = \frac{(-1)^n 2^{-n/2}}{n!}\, H_n\left(\frac{x}{\sqrt{2}}\right).$$

2 Charlier to Hermite

1. *Asymptotic expansion for large a:*

$$C_n(x,a) = B^n \sum_{k=0}^{n} \frac{c_k}{B^k}\, \frac{H_{n-k}(X)}{(n-k)!},$$

$$B = \sqrt{\frac{x}{2a^2}}, \qquad X = \frac{a-x}{\sqrt{2x}}.$$

2. *Coefficients:*

$$c_0 = 1, \quad c_1 = c_2 = 0$$

and, for $k = 3, 4, 5, \ldots$,

$$a^3 k c_k = a^2(k-1)c_{k-1} - x c_{k-3}.$$

3. *Asymptotic property:*

$$\frac{c_k}{B^k}\, H_{n-k}(X) = \mathcal{O}(a^{n-k}), \qquad a \to \infty.$$

4. *Limit (known):*

$$\lim_{a\to\infty} (2a)^{n/2} C_n(\sqrt{2a}x + a, a) = (-1)^n H_n(x).$$

3 Meixner-Pollaczeck to Hermite

1. *Asymptotic expansion for large λ:*

$$P_n^{(\lambda)}(x;\phi) = B^n \sum_{k=0}^{n} \frac{c_k}{B^k} \frac{H_{n-k}(X)}{(n-k)!},$$

$$B = \sqrt{-\lambda\cos(2\phi) - x\sin(2\phi)}, \quad X = \frac{\lambda\cos\phi + x\sin\phi}{B}.$$

2. *Coefficients:*

$$c_0 = 1, \quad c_1 = c_2 = 0, \quad c_3 = \frac{2}{3}[\lambda\cos(3\phi) + x\sin(3\phi)]$$

and, for $k = 4, 5, 6, ...,$

$$\begin{aligned} kc_k &= 2\cos\phi(k-1)c_{k-1} - (k-2)c_{k-2} + 2[\lambda\cos(3\phi) + x\sin(3\phi)]c_{k-3} \\ &\quad -2[\lambda\cos(2\phi) + x\sin(2\phi)]c_{k-4}. \end{aligned}$$

3. *Asymptotic property:*

$$\frac{c_k}{B^k} H_{n-k}(X) = \mathcal{O}(\lambda^{n/2+[k/3]-k}), \qquad \lambda \to \infty.$$

4. *Limit (known):*

$$\lim_{\lambda\to\infty} \lambda^{-n/2} P_n^{(\lambda)}\left(\frac{x\sqrt{\lambda} - \lambda\cos\phi}{\sin\phi};\phi\right) = \frac{1}{n!}H_n(x).$$

4 Jacobi to Hermite

1. *Asymptotic expansion for large $\alpha+\beta$:*

$$P_n^{(\alpha,\beta)}(x) = B^n \sum_{k=0}^{n} \frac{c_k}{B^k} \frac{H_{n-k}(X)}{(n-k)!},$$

$$X = \frac{2x + (\alpha+\beta)x + (\alpha-\beta)}{4B},$$

$$B = \sqrt{\frac{1}{2} - x^2 + \frac{1}{8}(\alpha+\beta) - \frac{1}{4}(\alpha-\beta)x - \frac{3}{8}x^2(\alpha+\beta)}.$$

2. *Coefficients:*

$$c_0 = 1, \quad c_1 = c_2 = 0, \quad c_3 = \frac{4}{3x^3} - x - \frac{1}{12}(\alpha-\beta) - \frac{1}{4}x(\alpha+\beta) + \frac{5}{12}x^3(\alpha+\beta) + \frac{1}{4}x^2(\alpha-\beta).$$

3. *Asymptotic property:*

$$\frac{c_k}{B^k} H_{n-k}(X) = \mathcal{O}((\alpha+\beta)^{n/2+[k/3]-k}), \qquad \alpha+\beta \to \infty, \quad \frac{\alpha-\beta}{\alpha+\beta} \to 0.$$

4. *Limit (new):*

$$\lim_{\alpha+\beta\to\infty} \frac{P_n^{(\alpha,\beta)}\left(\frac{\sqrt{2\alpha+2\beta}x+\beta-\alpha}{\alpha+\beta+2}\right)}{(\alpha+\beta)^{n/2}} = \frac{H_n(x)}{2^{3n/2}n!}, \text{ with } \frac{\alpha-\beta}{\alpha+\beta} \to 0.$$

5 Meixner to Hermite

1. *Asymptotic expansion for large β:*

$$M_n(x;\beta,c) = \frac{n!B^n}{(\beta)_n}\sum_{k=0}^{n}\frac{B_k}{z^k}\frac{H_{n-k}(X)}{(n-k)!},$$

$$B = \sqrt{\frac{1-c^2}{2c^2}x - \frac{\beta}{2}}, \quad X = \frac{(c-1)x+\beta c}{2cB}.$$

2. *Coefficients:*

$$c_0 = 1, \quad c_1 = c_2 = 0, \quad c_3 = \frac{(c^3-1)x + c^3\beta}{3c^3}$$

and, for $k = 4, 5, 6, ...,$

$$\begin{aligned} c^3 k c_k &= c^2(c+1)(k-1)c_{k-1} - c^2(k-2)c_{k-2} + (\beta c^3 + c^3 x - x)c_{k-3} \\ &\quad + (\beta c^2 + x - c^2 x)c_{k-4}. \end{aligned}$$

3. *Asymptotic property:*

$$\frac{c_k}{B^k}H_{n-k}(X) = \mathcal{O}(\beta^{n/2+[k/3]-k}), \qquad \beta \to \infty.$$

4. *Limit (new):*

$$\lim_{\beta\to\infty}(\beta)_n\left(\frac{2c}{\beta}\right)^{n/2} M_n\left(\frac{c\beta - x\sqrt{2c\beta}}{1-c};\beta,c\right) = H_n(x).$$

6 Krawtchouk to Hermite

1. *Asymptotic expansion for large N:*

$$\binom{N}{n}K_n(x;p,N) = B^n\sum_{k=0}^{n}\frac{c_k}{B^k}\frac{H_{n-k}(X)}{(n-k)!},$$

$$B = \sqrt{\frac{p^2N + x - 2xp}{2p^2}}, \quad X = \frac{Np - x}{2pB}.$$

2. *Coefficients:*

$$c_0 = 1, \quad c_1 = c_2 = 0, \quad c_3 = \frac{3xp - 3xp^2 + p^3N - x}{3p^3}$$

and, for $k = 4, 5, 6, ...,$

$$\begin{aligned} p^3 k c_k &= -p^2(2p-1)(k-1)c_{k-1} - p^2(p-1)(k-2)c_{k-2} \\ &+ \ (-x + 3xp - 3xp^2 + p^3N)c_{k-3} + (-2xp^2 + 3xp + p^3N - p^2N - x)c_{k-4}. \end{aligned}$$

3. *Asymptotic property:*

$$\frac{c_k}{B^k}H_{n-k}(X) = \mathcal{O}(N^{n/2+[k/3]-k}), \qquad N \to \infty.$$

4. *Limit (known):*

$$\lim_{N\to\infty}\binom{N}{n}\left(\sqrt{\frac{2p}{N(1-p)}}\right)^n K_n\left(pN + x\sqrt{2p(1-p)N};p,N\right) = \frac{(-1)^n}{n!}H_n(x).$$

7 Continuous Dual Hahn to Hermite

1. *Asymptotic expansion for large c:*

$$\frac{S_n^{a,b,c}(x^2)}{(a+b)_n n!} = \sum_{k=0}^{n} \frac{c_k B^{n-k}}{(n-k)!} H_{n-k}(X),$$

$$B = \sqrt{\frac{x^4 + [a(a+1) + b(b+1) + (a+b)^2]x^2 - ab[a(1+a) + b(1+b) + ab]}{2(a+b)^2(1+a+b)} - \frac{c}{2}},$$

$$X = \frac{c(a+b) + ab - x^2}{2B(a+b)}, \qquad f(w) = e^{-2BXw+B^2w^2}(1-w)^{-c+ix}{}_2F_1\left(\left.\begin{matrix} a+ix, b+ix \\ a+b \end{matrix}\right| w\right).$$

2. *Asymptotic property:*

$$\frac{c_k B^{n-k}}{(n-k)!} H_{n-k}(X) = \mathcal{O}\left(c^{n+\left[\frac{k}{3}\right]-k}\right) \text{ when } c \to \infty \text{ uniformly in } a, b \text{ with } \frac{a}{c} \text{ and } \frac{b}{c} \text{ bounded.}$$

3. *Limit:*

$$\lim_{a\to\infty} \frac{S_n^{a,b,ac}\left(\sqrt{ca^2 - xa\sqrt{2a(1+c)c}}\right)}{\left[a\left(x + \sqrt{a/2}\right)\sqrt{c(1+c)}\right]^n} = H_n(x).$$

8 Continuous Hahn to Hermite

1. *Asymptotic expansion for large a and b:*

$$\frac{(2a+2b-1)_n P_n^{\alpha,\beta,\overline{\alpha},\overline{\beta}}(x)}{(2a)_n(a+b+i(c-d))_n i^n} = \sum_{k=0}^{n} \frac{c_k B^{n-k}}{(n-k)!} H_{n-k}(X),$$

$$B = \sqrt{\frac{1}{2}p_1^2(x) - p_2(x)}, \qquad X = \frac{p_1(x)}{2B},$$

where

$$p_1(x) = \qquad i\,(1-2a-2b)\,\frac{bc + ad + x(a+b)}{a[a+b+i(c-d)]},$$

$$p_2(x) = \qquad (2a+2b-1)(a+b)\Bigg\{1 - \frac{(2a+2b+1)[a+i(c+x)]}{a[a+b+i(c-d)]}$$

$$+\frac{(2a+2b+1)(2a+2b+2)[a+i(c+x)][1+a+i(c+x)]}{2a(2a+1)[a+b+i(c-d))][a+b+1+i(c-d)]}\Bigg\},$$

$$f(w) = e^{-2BXw+B^2w^2}(1-w)^{1-2(a+b)}{}_3F_2\left(\left.\begin{matrix} a+b-\frac{1}{2}, a+b, a+i(c+x) \\ 2a, a+c+i(b-d) \end{matrix}\right| -\frac{4w}{(1-w)^2}\right).$$

2. *Asymptotic property:*

$$\frac{c_k B^{n-k}}{(n-k)!} H_{n-k}(X) = \mathcal{O}\left(a^{\frac{n-k}{2}+\left[\frac{k}{3}\right]}\right) \quad \text{when } a, b \to \infty \text{ with } a \sim b.$$

3. *Limit:*

$$\lim_{a\to\infty}\left[\left(-\frac{2}{i}\sqrt{\frac{a(1+b)}{b}}\right)^n P_n^{\alpha,\beta_1,\overline{\alpha},\overline{\beta_1}}\left(x\sqrt{\frac{ab}{1+b}}\right)\right] = H_n(x), \quad \text{with} \quad \beta_1 = ab + id.$$

9 Hahn to Hermite

1. *Asymptotic expansion for large a and N:*

$$\frac{(-N)_n Q_n^{a,b,N}(x)}{(b+1)_n n!} = \sum_{k=0}^{n} \frac{c_k B^{n-k}}{(n-k)!} H_{n-k}(X),$$

with B and X

$$B = \sqrt{\frac{1}{2}p_1^2(x) - p_2(x)}, \qquad X = \frac{p_1(x)}{2B},$$

and

$$\begin{aligned} p_2(x) &= (a+b+1)(a+b+2)\left[\frac{1}{2} - \frac{x(a+b+3)}{N(a+1)} + \frac{x(x-1)(a+b+3)(a+b+4)}{2N(N-1)(a+1)(a+2)}\right], \\ p_1(x) &= (a+b+1)\left[1 - \frac{x(a+b+2)}{N(a+1)}\right], \end{aligned}$$

$$f(w) = e^{-2BXw+B^2w^2}(1-w)^{-1-a-b} {}_3F_2\left(\left.\begin{matrix} (a+b+1)/2, (a+b+2)/2, -x \\ a+1, -N \end{matrix}\right| -\frac{4w}{(1-w)^2}\right).$$

2. *Asymptotic property:*

$$\frac{c_k B^{n-k}}{(n-k)!} H_{n-k}(X) = \mathcal{O}\left(N^{n+[\frac{k}{3}]-k}\right)$$

when a, $N \to \infty$ with $a \sim N$ uniformly in b with b/N bounded.

3. *Limit:*

$$\lim_{a\to\infty}\left[\left(\frac{2a}{2x+\sqrt{2a}}\right)^n \left(\frac{N(1+b)}{b(1+b+N)}\right)^{\frac{n}{2}} \times Q_n^{a,ab,aN}\left(\frac{aN(1+b) - x\sqrt{2abN(1+b)(1+b+N)}}{(1+b)^2}\right)\right] = H_n(x).$$

10 Dual Hahn to Hermite

1. *Asymptotic expansion for large N:*

$$\frac{(-N)_n R_n^{a,b,N}(\lambda(x))}{n!} = \sum_{k=0}^{n} \frac{c_k B^{n-k}}{(n-k)!} H_{n-k}(X), \quad \lambda(x) \equiv x(x+a+b+1),$$

$$B = \sqrt{\frac{N-x}{2} + \frac{x(x+b)[(2x+b-1)(1+a)+x(x+b)]}{2(a+1)^2(a+2)}}, \qquad X = \frac{\lambda(x) - N(a+1)}{2B(a+1)},$$

$$f(w) = e^{-2BXw+B^2w^2}(1-w)^{N-x} {}_2F_1\left(\left.\begin{matrix} -x, -b-x \\ a+1 \end{matrix}\right| w\right).$$

2. *Asymptotic property:*

$$\frac{c_k B^{n-k}}{(n-k)!} H_{n-k}(X) = \mathcal{O}\left(N^{n+[\frac{k}{3}]-k}\right) \text{ when } N \to \infty \text{ uniformly in } a, b \text{ with } \frac{a}{N} \text{ and } \frac{b}{N} \text{ bounded.}$$

3. *Limit:*

$$\lim_{a\to\infty}\left[\left(-\sqrt{2a}\right)^n R_n^{a,b,aN}\left(\frac{a}{2}\left[\sqrt{1+4N+4Nx\sqrt{2/a}}-1\right]\right)\right] = H_n(x).$$