1 Meixner-Pollaczeck to Charlier

1. *Asymptotic expansion for large λ:*

$$P_n^{(\lambda)}(x,\phi) = B^n \sum_{k=0}^{n} \frac{c_k}{B^k} \frac{C_{n-k}(X,A)}{(n-k)!},$$

$$X = -\frac{2\left(\lambda\cos(2\phi)+x\sin(2\phi)\right)^3}{\left(\lambda\cos(3\phi)+x\sin(3\phi)\right)^2}, \quad B = -\frac{2\left(\lambda^2+x^2\right)\sin^2\phi}{\lambda\cos(3\phi)+x\sin(3\phi)},$$

$$A = -\frac{2\left(\lambda^2+x^2\right)\sin^2\phi\left(\lambda\cos(2\phi)+x\sin(2\phi)\right)}{\left(\lambda\cos(3\phi)+x\sin(3\phi)\right)^2}.$$

2. *Coefficients:*

$$c_0 = 1, \quad c_1 = c_2 = c_3 = 0, \quad c_4 = -\frac{\left(\lambda^2+x^2\right)\sin^2\phi}{2\left(\lambda\cos(2\phi)+x\sin(2\phi)\right)}$$

and, for $k = 5, 6, 7, \ldots$,

$$kc_k = h_1(k-1)c_{k-1} - h_2(k-2)c_{k-2} + h_3(k-3)c_{k-3} + h_4 c_{k-4},$$

with

$$h_1 = \frac{\lambda\cos\phi + 2\lambda\cos(3\phi) + x\left[\sin\phi + 2\sin(3\phi)\right]}{\lambda\cos(2\phi)+x\sin(2\phi)}, \quad h_3 = \frac{\lambda\cos(3\phi)+x\sin(3\phi)}{\lambda\cos(2\phi)+x\sin(2\phi)},$$

$$h_2 = \frac{2\lambda\cos(2\phi)+\lambda\cos(4\phi)+8x\cos^3\phi\sin\phi}{\lambda\cos(2\phi)+x\sin(2\phi)}, \quad h_4 = -\frac{2\left(\lambda^2+x^2\right)\sin^2\phi}{\lambda\cos(2\phi)+x\sin(2\phi)}.$$

3. *Asymptotic property:*

$$\frac{c_k}{B^k} C_{n-k}(X,A) = \mathcal{O}(\lambda^{\lfloor k/4 \rfloor - k}), \qquad \lambda \to \infty.$$

4. *Limit (new):*

$$\lim_{\lambda\to\infty} \lambda^{-n} P_n^{(\lambda)}(x,\phi) = \frac{2^n}{n!}\left(2\cos(2\phi)-1\right)^n \cos^{2n}\phi\cos^{-n}(3\phi).$$

2 Jacobi to Charlier

1. *Asymptotic expansion for large β:*

$$P_n^{(\alpha,\beta)}(x) = B^n \sum_{k=0}^{n} \frac{c_k}{B^k} \frac{C_{n-k}(X,A)}{(n-k)!},$$

$$X = \frac{1}{4}\left\{4+\alpha+\beta-2x(\alpha-\beta)-x^2[8+3(\alpha+\beta)]\right\},$$

$$A = B = \frac{1}{4}(1-x)\left\{4+\alpha+\beta+2(\alpha-\beta)+x\left[8+3(\alpha+\beta)\right]\right\}.$$

2. *Coefficients:*

$$c_0 = 1, \quad c_1 = c_2 = 0,$$

$$c_3 = \frac{1}{3} + \frac{\beta}{6} - \left(1 + \frac{5\alpha}{12} + \frac{\beta}{12}\right) x - \left(\frac{2}{3} + \frac{\beta}{2}\right) x^2$$

$$+ \left(\frac{4}{3} + \frac{5}{12}(\alpha + \beta)\right) x^3,$$

$$c_4 = \frac{1}{2} + \frac{7}{64}(-\alpha + \beta) + \frac{5}{16}(-\alpha + \beta)\, x - \left(\frac{5}{2} + \frac{21}{32}(\alpha + \beta)\right) x^2$$

$$+ \frac{5}{16}(\alpha - \beta)\, x^3 + \left(2 + \frac{35}{64}(\alpha + \beta)\right) x^4.$$

3. *Asymptotic property:*

$$\frac{c_k}{B^k} C_{n-k}(X, A) = \mathcal{O}(\beta^{[k/3]-k}), \qquad \beta \to \infty.$$

4. *Limit (new):*

$$\lim_{\alpha+\beta\to\infty} (\alpha + \beta)^{-n} P_n^{(\alpha,\beta)}(x) = \frac{1}{n!}\left(\frac{x}{2}\right)^n, \text{ with } (\alpha - \beta)/(\alpha + \beta) \to 0.$$

3 Meixner to Charlier

1. *Asymptotic expansion for large β:*

$$M_n(x, \beta, c) = \frac{B^n n!}{(\beta)_n} \sum_{k=0}^{n} \frac{c_k}{B^k} \frac{C_{n-k}(X, A)}{(n-k)!},$$

$$X = -\frac{\left(\beta c^2 + (c^2 - 1)x\right)^3}{\left(\beta c^3 + (c^3 - 1)x\right)^2}, \qquad B = -\frac{(c-1)^2 x(\beta + x)}{\beta c^3 + (c^3 - 1)x},$$

$$A = -\frac{(c-1)^2 c x(\beta + x)\left(\beta c^2 + (c^2 - 1)x\right)}{\left(\beta c^3 + (c^3 - 1)x\right)^2}.$$

2. *Coefficients:*

$$c_0 = 1, \quad c_1 = c_2 = c_3 = 0, \quad c_4 = -\frac{(c-1)^2 x(\beta + x)}{4c^2(\beta c^2 + (c^2 - 1)x)}$$

and, for $k = 5, 6, 7 \ldots$,

$$k c_k = h_1(k-1)c_{k-1} + h_2(k-2)c_{k-2} + h_3(k-3)c_{k-3} + h_4 c_{k-4},$$

with

$$h_1 = \frac{\beta c^2(1 + 2c) - (2 + c - c^2 - 2c^3)x}{\beta c^3 + c(c^2 - 1)x}, \qquad h_3 = \frac{\beta c^3 - x + c^3 x}{\beta c^4 + c^2(c^2 - 1)x},$$

$$h_2 = \frac{(1-c)(1+c)^3 x - \beta c^3(2 + c)}{\beta c^4 + c^2(c^2 - 1)x}, \qquad h_4 = -\frac{(c-1)^2 x(\beta + x)}{\beta c^4 + c^2(c^2 - 1)x}.$$

3. *Asymptotic property:*

$$\frac{c_k}{B^k} C_{n-k}(X, A) = \mathcal{O}(\beta^{n-k}), \qquad \beta \to \infty.$$

4. *Limit (known):*

$$\lim_{\beta\to\infty} M_n\left(x, \beta, \frac{a}{a+\beta}\right) = C_n(x, a).$$

4 Krawtchouk to Charlier

1. *Asymptotic expansion for large N:*

$$\binom{N}{n} K_n(x,p,N) = B^n \sum_{k=0}^{n} \frac{c_k}{B^k} \frac{C_{n-k}(X,A)}{(n-k)!},$$

$$X = \frac{\left(Np^2+x-2px\right)^3}{\left(Np^3+(-3p^2+3p-1)x\right)^2}, \qquad B = \frac{(p-1)(N-x)x}{Np^3+(3p-3p^2-1)x},$$

$$A = -\frac{(p-1)p(N-x)x\left(Np^2+x-2px\right)}{\left(Np^3+(-3p^2+3p-1)x\right)^2}.$$

2. *Coefficients:*

$$c_0 = 1, \quad c_1 = c_2 = c_3 = 0, \quad c_4 = \frac{(p-1)^2 x(x-N)}{4p^2(Np^2+x-2px)}$$

and, for $k = 5, 6, 7, ...,$

$$kc_k = h_1(k-1)c_{k-1} + h_2(k-2)c_{k-2} + h_3(k-3)c_{k-3} + h_4 c_{k-4},$$

with

$$h_1 = \frac{Np^2(1-3p)+(7p^2-7p+2)x}{p(Np^2+x-2px)}, \qquad h_2 = \frac{(2p-1)^3 x - Np^3(3p-2)}{p^2(Np^2+x-2px)},$$

$$h_3 = \frac{(p-1)[(3p^2-3p+1)x-Np^3]}{p^2(Np^2+x-2px)}, \qquad h_4 = \frac{(p-1)^2 x(x-N)}{p^2(Np^2+x-2px)}.$$

3. *Asymptotic property:*

$$\frac{c_k}{B^k} C_{n-k}(X,A) = \mathcal{O}(N^{n-k}), \qquad N \to \infty.$$

4. *Limit (known):*

$$\lim_{N\to\infty} K_n\left(x, \frac{a}{N}, N\right) = C_n(x,a).$$

5 Continuous Dual Hahn to Charlier

1. *Asymptotic expansion for large a and b:*

$$\frac{S_n^{a,b,c}(x^2)}{(a+b)_n n!} = \sum_{k=0}^{n} \frac{c_k A^{n-k}}{(n-k)!} C_{n-k}(X,A),$$

with

$$A = p_1(x) + p_1^2(x) - 2p_2(x), \qquad X = p_1^2(x) - 2p_2(x),$$

$$p_1(x) = c + \frac{ab-x^2}{a+b},$$

$$p_2(x) = \frac{c(c+1)}{2} + \frac{abc}{a+b} + \frac{ab(1+a)(1+b) - [c+2ab+(1+c)(1+2(a+b))]x^2 + x^4}{2(a+b)(1+a+b)},$$

and

$$f(w) = e^{-Aw}(1-w)^{-X-c+ix} {}_2F_1\left(\left.\begin{matrix} a+ix, b+ix \\ a+b \end{matrix}\right| w\right).$$

2. *Asymptotic property:*

$$\frac{c_k A^{n-k}}{(n-k)!} C_{n-k}(X, A) = \mathcal{O}\left(a^{n+[\frac{k}{3}]-k}\right)$$

when $a, b \to \infty$ uniformly in c with $a \sim b$ and c/a bounded.

3. *Limit:*

$$\lim_{a\to\infty} \frac{S_n^{a,ab,\tilde{c}}(\tilde{x})}{c^n a^n (1+b)^n} = C_n(x, c),$$

where

$$\begin{aligned}
\tilde{x} &= \frac{1}{2}\left[-a^2(1+b^2) + a(1+b)\sqrt{a^2(1-b)^2 + 4ac(1+b)}\right], \\
\tilde{c} &= \frac{1}{2}\left[2c - 2x - a(1-b) + \sqrt{a^2(1-b)^2 + 4ac(1+b)}\right].
\end{aligned}$$

6 Continuous Hahn to Charlier

1. *Asymptotic expansion for large a and b:*

$$\frac{(2a+2b-1)_n P_n^{\alpha,\beta,\overline{\alpha},\overline{\beta}}(x)}{(2a)_n (a+b+i(c-d))_n i^n} = \sum_{k=0}^{n} \frac{c_k A^{n-k}}{(n-k)!} C_{n-k}(X, A),$$

$$A = p_1(x) + p_1^2(x) - 2p_2(x), \qquad X = p_1^2(x) - 2p_2(x),$$

$$p_1(x) = i\,(1-2a-2b)\,\frac{bc+ad+x(a+b)}{a[a+b+i(c-d)]},$$

$$\begin{aligned}
p_2(x) = {} & (2a+2b-1)(a+b)\Big\{1 - \frac{(2a+2b+1)[a+i(c+x)]}{a[a+b+i(c-d)]} \\
& + \frac{(2a+2b+1)(2a+2b+2)[a+i(c+x)][1+a+i(c+x)]}{2a(2a+1)[a+b+i(c-d))][a+b+1+i(c-d)]}\Big\},
\end{aligned}$$

$$f(w) = e^{-Aw}(1-w)^{-X+1-2(a+b)} {}_3F_2\left(\begin{matrix} a+b-\frac{1}{2}, a+b, a+i(c+x) \\ 2a, a+c+i(b-d) \end{matrix} \,\middle|\, -\frac{4w}{(1-w)^2}\right).$$

2. *Asymptotic property:*

$$\frac{c_k A^{n-k}}{(n-k)!} C_{n-k}(X, A) = \mathcal{O}\left(b^{[\frac{n-k+1}{2}]+[\frac{k}{3}]}\right) \quad \text{when} \quad a,\, b \to \infty \quad \text{with} \quad a \sim b.$$

7 Hahn to Charlier

1. *Asymptotic expansion for large a, b and N:*

$$\frac{(a+b+1)_n Q_n^{a,b,N}(x)}{n!} = \sum_{k=0}^{n} \frac{c_k A^{n-k}}{(n-k)!} C_{n-k}(X, A),$$

with A and X

$$A = p_1(x) + p_1^2(x) - 2p_2(x), \qquad X = p_1^2(x) - 2p_2(x),$$

and

$$\begin{aligned}
p_2(x) &= (a+b+1)(a+b+2)\left[\frac{1}{2} - \frac{x(a+b+3)}{N(a+1)} + \frac{x(x-1)(a+b+3)(a+b+4)}{2N(N-1)(a+1)(a+2)}\right], \\
p_1(x) &= (a+b+1)\left[1 - \frac{x(a+b+2)}{N(a+1)}\right],
\end{aligned}$$

$$f(w) = e^{-Aw}(1-w)^{-X-a-b-1}{}_3F_2\left(\left.\begin{array}{c}(a+b+1)/2,(a+b+2)/2,-x\\ a+1,-N\end{array}\right| -\frac{4w}{(1-w)^2}\right).$$

2. *Asymptotic property:*

$$\frac{c_k A^{n-k}}{(n-k)!}C_{n-k}(X,A) = \mathcal{O}\left(a^{n-k}\right) \quad \text{when} \quad a,b,N\to\infty \quad \text{with} \quad a\sim b\sim N.$$

8 Dual Hahn to Charlier

1. *Asymptotic expansion for large a and N:*

$$\frac{(-N)_n R_n^{a,b,N}(\lambda(x))}{n!} = \sum_{k=0}^{n}\frac{c_k A^{n-k}}{(n-k)!}C_{n-k}(X,A), \qquad \lambda(x)\equiv x(x+a+b+1),$$

$$A = p_1(x)+p_1^2(x)-2p_2(x), \qquad X = p_1^2(x)-2p_2(x),$$

$$\begin{aligned} p_1(x) &= \frac{\lambda(x)}{a+1}-N,\\ p_2(x) &= \frac{(N-x)(N-x-1)}{2}+\frac{x(b+x)\left[2(2+a)(x-N)+(b+x-1)(x-1)\right]}{2(a+1)(2+a)} \end{aligned}$$

$$f(w) = e^{-Aw}(1-w)^{-X+N-x}{}_2F_1\left(\left.\begin{array}{c}-x,-b-x\\ a+1\end{array}\right| w\right).$$

2. *Asymptotic property:*

$$\frac{c_k A^{n-k}}{(n-k)!}C_{n-k}(X,A) = \mathcal{O}\left(a^{n-k-2}\right) \quad \text{when} \quad a,N\to\infty \quad \text{with} \quad a\sim N.$$