Monografías Matemáticas *García de Galdeano* **41**, 31–40 (2018)

Steady-state analysis of the single vacation $PH/MSP/1/\infty$ queue using roots

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Abstract. We consider an infinite-buffer single-server queue where inter-arrival times are phase-type (PH), the service is provided according to Markovian service process (MSP), and the server may take single, exponentially distributed vacations when the queue is empty. The proposed analysis is based on roots of the associated characteristic equation of the vector-generating function (VGF) of system-length distribution at a pre-arrival epoch. At the end, we present numerical results in the form of a table to show the effect of model parameters on the system-length distribution at a pre-arrival epoch.

Keywords: Markovian arrival process (MAP); Markovian service process (MSP); renewal input; infinite-buffer; exponential single vacation.

AMS classification: 60K25; 90B22; 68M20.

§1. Introduction

In recent times, queueing models with non-renewal arrival and service processes have been used to model networks of complex computer and communication systems. Traditional queueing analysis using Poisson processes is not powerful enough to capture the correlated nature of arrival (service) processes. The performance analysis of correlated type of arrival processes may be done through some analytically tractable arrival process viz., Markovian arrival process (MAP), see Lucantoni et al. [12]. The MAP has the property of both time varying arrival rates and correlation between inter-arrival times. To consider batch arrivals of variable capacity, Lucantoni [11] introduced batch Markovian arrival process ($BMAP$). The processes MAP and $BMAP$ are convenient representations of a versatile Markovian point process, see Neuts [13] and Ramaswami [14]. Like the MAP, Markovian service process (MSP) is a versatile service process which can capture the correlation among the successive service times. Several other service processes, e.g., Poisson process, Markov modulated Poisson process ($MMPP$) and phase-type (PH) renewal process can be considered as special cases of MSP. For details of MSP, the readers are referred to Bocharov [3] and Albores and Tajonar [1]. The analysis of finite-buffer $G/MSP/1/r$ ($r \leq \infty$) queue has been performed by Bocharov et al. [4]. Queueing with multiple servers such as $GI/MSP/c/r$ has been analyzed by Albores and Tajonar [1]. Gupta and Banik [10] analyzed $GI/MSP/1$ queue with finite- as well as infinite-buffer capacity using a combination of embedded Markov chain and supplementary variable method.

In this paper, we carry out the analysis of the $PH/MSP/1/\infty$ queue with exponential single vacation through the calculation of roots of the denominator of the underlying vector generating function of the steady-state probabilities at pre-arrival epoch (see Equations (20) and (21)). In this connection, the readers are referred to Chaudhry et al. [7, 8, 6], Tijms [16] and Chaudhry et al. [9] who have used the roots method. The roots can be easily found using one of the several commercially available packages such as Maple and Mathematica. The algorithm for finding such roots is available in some papers, e.g., see Chaudhry et al. [9]. The purpose of studying this queueing model using roots is that we obtain computationally simple and analytically closed form solution to the infinite-buffer $PH/MSP/1$ queue with the vacation time following exponential distribution. It may be remarked here that the matrix-geometric method (MGM) uses iterative procedure to get steady-state probabilities at the pre-arrival epochs. Further, it is well known that for the case of the MGM it is required to solve the non-linear matrix equation with the dimension of each matrix in this equation being the number of service-phases involved in a $PH/MSP/1$ queue. In the case of the roots method, we do not have to investigate the structure of the transition probability matrices (TPM) at the embedded pre-arrival epochs. It may be mentioned here that the basic idea of correlated service was first introduced by Chaudhry [5]. Further, it may be remarked here that the analysis of the infinite-buffer queues with renewal input and exponential service time under exponential server vacation(s) has been carried out by Tian and Zhang [15], see Chapter 4. The queueing model that we are going to consider has non-renewal service (MSP) and exponential single vacation time. Finally, it may be remarked here that we perform steady-state analysis of a stable $PH/MSP/1$ queueing system, i.e., under the assumption that the traffic intensity (which is the mean number of customer arrival during an average service time) is strictly less than one. Obviously this assumption is not a practical one because many queueing situations face instability due to the traffic intensity is more than or very close to one. One may be interested in the analysis of such type of instable queues in future using a different methodology than the one used in this paper. Therefore, further research in the close to instability domain is necessary.

Contents. The paper is organized as follows. In Section 2 we give the description of the model. Section 3 computes the vector generating function of the number of customers served during an inter-arrival time (see Equation (9)), a crucial quantity which applies to the model without vacation as well, and whose computation is the main difficulty. Section 4 concerns the model with vacations, whose effect is quantized in (12), and gives the vector generating function of the steady-state probabilities at pre-arrival epochs (see equation (17), where (12) intervenes in the numerator). This section further details the application of the roots method in our case. In section 5 numerical results have been presented. In addition, important performance measure (mean system-length and mean sojourn time), system-length distributions at arbitrary- and post-departure-epoch, and expected busy and idle periods are discussed in an internet supplement see Banik et al. [2].

§2. Model description

We consider a single-server infinite-buffer queueing system with the server's single vacation. The inter-arrival time of customers, the service time of a customer and the vacation time of the server are represented by the generic random variables (r.v.'s) A, S, and V, respectively. Let $F_X(x)$ denote the distribution function (D. F.) of the random variable X with $f_X(x)$ and $f_X^*(s)$ the corresponding probability density function (p.d.f.) and Laplace-Stieltjes transform

(LST), respectively. The inter-arrival time A is assumed to have a general distribution with p.d.f. $f_A(x)$, D. F. $F_A(x)$ and LST $f_A^*(s)$.

Arrivals. The inter-arrival times are assumed to be independent and identically-distributed (i.i.d.) random variables and they are independent of the service process as well as vacation time. The inter-arrival time distribution PH is an important special case of general distribution as the distribution possesses nice vector and matrix form representation. Several probability distributions such as Earlang, hyper-exponential, generalized Earlang, Coxian etc. can be treated as special cases of PH-distribution. It may be noted here that PH-distribution is a special case of general distribution. If the inter-arrival times follow PH-type distribution with irreducible representation $(\boldsymbol{\alpha}, \boldsymbol{T})$, where $\boldsymbol{\alpha}$ and $\boldsymbol{T}$ are a vector and a matrix of dimension $1 \times \eta$ and $\eta \times \eta$, respectively, the p.d.f. and D.F. of inter-arrival times are given by

$$F_A(x) = 1 - \boldsymbol{\alpha}\, e^{\boldsymbol{T}x}\boldsymbol{e}_\eta, \quad \text{for } x \geq 0, \tag{1}$$

$$\text{and} \quad f_A(x) = -\boldsymbol{\alpha}\, e^{\boldsymbol{T}x}\boldsymbol{T}\boldsymbol{e}_\eta = \boldsymbol{\alpha}\, e^{\boldsymbol{T}x}\boldsymbol{T}^0, \quad \text{for } x > 0, \tag{2}$$

where $\boldsymbol{T}^0$ is a non-negative vector and satisfies $\boldsymbol{T}\mathbf{e}_\eta + \boldsymbol{T}^0 = \mathbf{0}$ and $\boldsymbol{e}_\eta$ is an $\eta \times 1$ vector with all its elements equal to 1. Throughout the paper we write a subscript as the dimension of the column vector $\boldsymbol{e}$ and sometimes we write $\boldsymbol{e}$ by dropping its subscript. The mean inter-arrival time during a normal busy period is given by

$$\frac{1}{\lambda} = \boldsymbol{\alpha} \int_0^\infty x e^{\boldsymbol{T}x}\, dx(-\boldsymbol{T})\boldsymbol{e}_\eta = -\boldsymbol{\alpha}(\boldsymbol{T})^{-1}\mathbf{e}_\eta. \tag{3}$$

Services. The customers are served singly according to the continuous-time Markovian service process (MSP) with matrix representation $(\mathbf{L}_0, \boldsymbol{L}_1)$. The MSP is a generalization of the Poisson process where the services are governed by an underlying m-state Markov chain. For more details on MSP, the readers are referred to recent papers by Chaudhry et al. [7, 6]. Let $N(t)$ denote the number of customers served in t units of time and $J(t)$ the state of the underlying Markov chain at time t with its state space $\{i : 1 \leq i \leq m\}$. Then $\{N(t), J(t)\}$ is a two-dimensional Markov process with state space $\{(\ell, i) : \ell \geq 0, 1 \leq i \leq m\}$. Average service rate of customers $\mu^\star$ (the so called fundamental service rate) of the stationary MSP is given by $\mu^\star = \overline{\boldsymbol{\pi}}\boldsymbol{L}_1\boldsymbol{e}$, where $\overline{\boldsymbol{\pi}} = [\overline{\pi}_1, \overline{\pi}_2, \ldots, \overline{\pi}_m]$ with $\overline{\pi}_j$ denoting the steady-state probability of servicing a customer in phase j $(1 \leq j \leq m)$. The stationary probability row-vector $\overline{\boldsymbol{\pi}}$ can be calculated from $\overline{\boldsymbol{\pi}}\boldsymbol{L} = \mathbf{0}$ with $\overline{\boldsymbol{\pi}}\boldsymbol{e} = 1$, where $\boldsymbol{L} = \boldsymbol{L}_0 + \boldsymbol{L}_1$. The customers are served singly according to a MSP with steady-state mean service time $1/\mu^*$. The traffic intensity is given by $\rho = \lambda E(S) = \lambda/\mu^*$ which is independent of the vacation process.

Vacations. The server is allowed to take a single vacation whenever the system becomes empty. On return from a vacation if the server finds the system nonempty he will serve the customers present in the queue, otherwise the server waits for a customer to arrive and the system continues in this manner. For an exponential single vacation time represented by the r.v. V, the LST, p.d.f. and D.F. are given as follows:

$$f_V^*(s) = \frac{\gamma}{\gamma + s}, \quad f_V(x) = \gamma e^{-\gamma x}, \quad F_V(x) = 1 - e^{-\gamma x}, \tag{4}$$

where $1/\gamma$ (> 0) is assumed as the mean vacation time. The vacation times are independent of the inter-arrival time as well as of the service processes.

§3. The vector generating function of the number of customers served during an inter-arrival

Let us denote by $\{\boldsymbol{P}(n,t) : n \geq 0, t \geq 0\}$ as the $m \times m$ matrix whose (i,j)th element is the conditional probability defined as $P_{i,j}(n,t) = Pr\{N(t) = n, J(t) = j | N(0) = 0, J(0) = i\}, \quad 1 \leq i,j \leq m$. Using probabilistic arguments, we obtain the following system of matrix differential-difference equations:

$$\frac{d}{dt}\boldsymbol{P}(0,t) = \boldsymbol{P}(0,t)\boldsymbol{L}_0 \tag{5}$$

$$\frac{d}{dt}\boldsymbol{P}(n,t) = \boldsymbol{P}(n,t)\boldsymbol{L}_0 + \boldsymbol{P}(n-1,t)\boldsymbol{L}_1, \ n \geq 1, \tag{6}$$

with $\boldsymbol{P}(0,0) = \boldsymbol{I}_m$ and $\boldsymbol{P}(n,0) = \boldsymbol{0}$, $n \geq 1$, where $\boldsymbol{I}_m$ is the identity matrix of order $m \times m$. Let us define the matrix-generating function $\boldsymbol{P}^*(z,t)$ as

$$\boldsymbol{P}^*(z,t) = \sum_{n=0}^{\infty} \boldsymbol{P}(n,t)z^n = e^{\boldsymbol{L}(z)t}, \quad |z| \leq 1, \tag{7}$$

see Chaudhry et al. [6], where $\boldsymbol{L}(z) = \boldsymbol{L}_0 + \boldsymbol{L}_1 z$.

Let $\boldsymbol{S}(n)$ $(n \geq 0)$ denote the matrix of order $m \times m$ whose (i,j)th element represents the conditional probability that during an inter-arrival period n customers are served and the service process passes to phase j, provided at the initial instant of the previous arrival epoch there were at least n customers in the system and the service process was in phase i. Then

$$\boldsymbol{S}(n) = \int_0^{\infty} \boldsymbol{P}(n,t) dF_A(t), \ n \geq 0. \tag{8}$$

If $\boldsymbol{S}(z)$ is the matrix-generating function of $\boldsymbol{S}(n)$, and $S_{i,j}(z)$ $(1 \leq i,j \leq m)$ are the elements of $\boldsymbol{S}(z)$, then multiplying (8) by z^n, $|z| \leq 1$ and summing from $n = 0$ to ∞, we get

$$\begin{aligned}\boldsymbol{S}(z) &= \sum_{n=0}^{\infty} \boldsymbol{S}(n)z^n = \lim_{N\to\infty} \sum_{n=0}^{N} \int_0^{\infty} \boldsymbol{P}(n,t)z^n dF_A(t) = \int_0^{\infty} \lim_{N\to\infty} \sum_{n=0}^{N} \boldsymbol{P}(n,t)z^n dF_A(t) \\ &= \int_0^{\infty} \boldsymbol{P}^*(z,t) dF_A(t) = \int_0^{\infty} e^{\boldsymbol{L}(z)t} f_A(t)dt = f_A^*(-\boldsymbol{L}(z)),\end{aligned} \tag{9}$$

where we have interchanged limit and integration on the first lign using element-wise the dominated convergence theorem and the fact that $dF_A(t)$ is a probability measure. Indeed, it suffices to check for the (i,j)-th's component that

$$|\sum_{n=0}^{N} P_{i,j}(n,t)z^n| \leq \sum_{n=0}^{\infty} P_{i,j}(n,t) = P_{i,j}[N(t) < \infty] \leq 1.$$

When inter-arrival time distributions are of PH-type having the representation $(\boldsymbol{\alpha}, \boldsymbol{T})$, $\boldsymbol{S}(z)$ is given by [8]:

$$\boldsymbol{S}(z) = (\boldsymbol{I}_m \otimes \boldsymbol{\alpha})(\boldsymbol{L}(z) \oplus \boldsymbol{T})^{-1}(\boldsymbol{I}_m \otimes \boldsymbol{T}\boldsymbol{e}_\eta), \tag{10}$$

with $\boldsymbol{L}(z) \oplus \boldsymbol{T} = (\boldsymbol{L}(z) \otimes \boldsymbol{I}_v) + (\boldsymbol{I}_m \otimes \boldsymbol{T})$, where $\oplus$ and $\otimes$ are used for Kronecker product and sum, respectively. Knowing that each element of $\boldsymbol{L}(z)$ is a polynomial in z, each element of $\boldsymbol{L}(z) \oplus \boldsymbol{T}$ is also a polynomial in z and hence the determinant of $(\boldsymbol{L}(z) \oplus \boldsymbol{T})$ is a rational function in z. Thus, from the above expression for $\boldsymbol{S}(z)$, we can immediately say that each element of $\boldsymbol{S}(z)$ is a rational function in z with the same denominator. One may note that in case the degree of the polynomials in each element of $\boldsymbol{S}(z)$ is very high, it may be difficult or time consuming to calculate the roots of the characteristic equation

$$det[\boldsymbol{I}_m z - \boldsymbol{S}(z)] = 0. \tag{11}$$

This difficulty may be minimized by calculating $\boldsymbol{S}(z)$ via $\boldsymbol{S}(z) = \lim_{N\to\infty} \sum_{n=0}^{N} \boldsymbol{S}_n z^n$, where $\boldsymbol{S}(n)$ may be obtained as proposed in [6, 11].

§4. Stationary system-length distribution of $GI/MSP/1/\infty$ queue at pre-arrival epoch

We consider a $GI/MSP/1/\infty$ queueing system with single vacation as described above. In the following subsections we obtain steady-state distributions for this queueing system at different epochs considering $\rho < 1$.

Let ω denote the probability that the remaining vacation time ($\widehat{V}$, say) exceeds an inter-arrival time A, then

$$\begin{aligned}\omega &= \int_0^\infty Pr(A < \widehat{V} | A = x).f_A(x)\,dx = \int_0^\infty Pr(\widehat{V} > x).f_A(x)\,dx \\ &= \int_0^\infty (1 - F_{\widehat{V}}(x)) f_A(x)\,dx = f_A^*(\gamma).\end{aligned} \tag{12}$$

Remark 1. Note similarity of this equation with (9).

Consider the system just before arrival epochs which are taken as embedded points. Let $t_0, t_1, t_2, ...$ be the time epochs at which arrivals occur and t_k^- the time instant before t_k. The inter-arrival times $T_{k+1} = t_{k+1} - t_k$, $k = 0, 1, 2, \ldots$ are i.i.d.r.v.'s with common distribution function $F_A(x)$. The state of the system at t_k^- is defined as $\zeta_k = \{N_{t_k^-}, J_{t_k^-}, \xi_{t_k^-}\}$ where $N_{t_k^-}$ is the number of customers n (≥ 0) present in the system including the one currently in service. Whereas $J_{t_k^-} = \{j\}$, $1 \leq j \leq m$, denotes phase of the service process and $\xi_{t_k^-} = l = 0$ or 1 indicates that the server is on vacation ($l = 0$) or busy ($l = 1$). In the limiting case, we define the following probabilities:

$$\pi_{j,0}^-(n) = \lim_{k\to\infty} P\{N_{t_k^-} = n,\ J_{t_k^-} = j,\ \xi_{t_k^-} = 0\},\ n \geq 0,\ 1 \leq j \leq m,$$

$$\pi_{j,1}^-(n) = \lim_{k\to\infty} P\{N_{t_k^-} = n,\ J_{t_k^-} = j,\ \xi_{t_k^-} = 1\},\ n \geq 0,\ 1 \leq j \leq m.$$

Let $\boldsymbol{\pi}_0^-(n)$ and $\boldsymbol{\pi}_1^-(n)$ be the row vectors of order $1 \times m$ whose j-th components are $\pi_{j,0}^-(n)$ and $\pi_{j,1}^-(n)$, respectively.

Observing the state of the system at two consecutive embedded points, we have an embedded Markov chain whose state space is equivalent to $\Omega = \{(k, j, 0),\ k \geq 0, 1 \leq j \leq$

$m,\} \cup \{(n, j, 1),\ n \geq 1,\ 1 \leq j \leq m\}$. The one-step transition probability matrix (TPM) of the above Markov chain $\mathcal{P}$ may be obtained. To obtain the vector-generating function (VGF) of the distribution of the number of customers in the system at pre-arrival epochs, we write $\boldsymbol{\pi}^- = \boldsymbol{\pi}^-\mathcal{P}$, where
$\boldsymbol{\pi}^-=[\boldsymbol{\pi}_0^-(0), \boldsymbol{\pi}_0^-(1), \boldsymbol{\pi}_1^-(1), \boldsymbol{\pi}_0^-(2), \boldsymbol{\pi}_1^-(2), \cdots\cdots]$. Therefore, we have the following system of vector difference equations

$$\boldsymbol{\pi}_0^-(0) = \sum_{k=0}^{\infty} \boldsymbol{\pi}_0^-(k)\Big((1-\omega)\boldsymbol{V}_{k+1}^* - \boldsymbol{C}_{k+1}^*\Big) + \sum_{n=0}^{\infty} \boldsymbol{\pi}_1^-(n)\Big((1-\omega)\boldsymbol{V}_{n+1}^*\Big), \tag{13}$$

$$\boldsymbol{\pi}_0^-(n) = \boldsymbol{\pi}_0^-(n-1)\omega \boldsymbol{I}_m, \quad n \geq 1, \tag{14}$$

$$\boldsymbol{\pi}_1^-(0) = \sum_{k=0}^{\infty} \boldsymbol{\pi}_0^-(k)\boldsymbol{C}_{k+1}^* + \sum_{n=0}^{\infty} \boldsymbol{\pi}_1^-(n)\Big(\omega(\boldsymbol{V}_{n+1}^* + \sum_{j=0}^{n} \boldsymbol{V}_j) + (1-\omega)\sum_{j=0}^{n} \boldsymbol{V}_j - \sum_{i=0}^{n} \boldsymbol{S}_i\Big), \tag{15}$$

$$\boldsymbol{\pi}_1^-(n) = \sum_{k=n-1}^{\infty} \boldsymbol{\pi}_0^-(k)(1-\omega)\boldsymbol{V}_{k-n+1} + \sum_{j=n-1}^{\infty} \boldsymbol{\pi}_1^-(j)\boldsymbol{S}_{j-n+1}, \quad n \geq 1, \tag{16}$$

where the matrices $\boldsymbol{V}_j\ (j \geq 0)$, $\boldsymbol{V}_k^*\ (k \geq 1)$, $\boldsymbol{C}_l^*\ (l \geq 1)$ may be obtained analogous to $\boldsymbol{S}_k^*\ (k \geq 1)$ defined in [6], and ω is defined as the probability that remaining vacation time $(\widehat{V})$ exceeds an inter-arrival time (A), see Equation (12). Multiplying (16) by z^n, summing from $n = 1$ to ∞, after adding (15) and using the vector-generating function $\boldsymbol{\pi}_1^{-*}(z) = \sum_{n=0}^{\infty} \boldsymbol{\pi}_1^-(n)z^n$, we obtain

$$\boldsymbol{\pi}_1^{-*}(z) = \frac{\Big(\sum_{j=0}^{\infty}\sum_{i=j+1}^{\infty} \boldsymbol{\pi}_0^-(j)(1-\omega)\boldsymbol{V}_i z^{j-i+1} + \boldsymbol{Y}\Big)Ad[\boldsymbol{I}_m - z\boldsymbol{S}(z^{-1})]}{det[\boldsymbol{I}_m - z\boldsymbol{S}(z^{-1})]}, \tag{17}$$

where $\boldsymbol{Y} = \sum_{k=0}^{\infty} \boldsymbol{\pi}_0^-(k)\boldsymbol{C}_{k+1}^* + \sum_{n=0}^{\infty} \boldsymbol{\pi}_1^-(n)\Big(\omega(\boldsymbol{V}_{n+1}^* + \sum_{j=0}^{n} \boldsymbol{V}_j) + (1-\omega)\sum_{j=0}^{n} \boldsymbol{V}_j - \sum_{i=0}^{n} \boldsymbol{S}_i\Big)$ and $Ad[\boldsymbol{I}_m - z\boldsymbol{S}(z^{-1})]$ is the adjoint of the matrix $[\boldsymbol{I}_m - z\boldsymbol{S}(z^{-1})]$. For further analysis, we first determine an analytic expression for each component of $\boldsymbol{\pi}_1^{-*}(z)$. Each component $\pi_{j,1}^{-*}(z)$ defined as $\pi_{j,1}^{-*}(z) = \sum_{n=0}^{\infty} \pi_{j,1}^-(n)z^n$ of the VGF $\boldsymbol{\pi}_1^{-*}(z)$ given in (17) being convergent in $|z| \leq 1$ implies that $\boldsymbol{\pi}_1^{-*}(z)$ is convergent in $|z| \leq 1$. As each element of $\boldsymbol{S}(z^{-1})$ is a rational function, (see Chaudhry et al. [6]) each element of $det[\boldsymbol{I}_m - z\boldsymbol{S}(z^{-1})]$ is also a rational function and we assume that $det[\boldsymbol{I}_m - z\boldsymbol{S}(z^{-1})] = \frac{d(z)}{\varphi(z)}$. Equation (17) can be rewritten element-wise as

$$\pi_{j,1}^{-*}(z) = \frac{\xi_j(z)}{d(z)}, \quad 1 \leq j \leq m, \tag{18}$$

where $\xi_j(z)$ is the j-th component of $\Big(\sum_{j=0}^{\infty}\sum_{i=j+1}^{\infty} \boldsymbol{\pi}_0^-(j)(1-\omega)\boldsymbol{V}_i z^{j-i+1} + \sum_{k=0}^{\infty} \boldsymbol{\pi}_0^-(k)\boldsymbol{C}_{k+1}^*$ $+\sum_{n=0}^{\infty} \boldsymbol{\pi}_1^-(n)\Big(\omega(\boldsymbol{V}_{n+1}^* + \sum_{j=0}^{n} \boldsymbol{V}_j) + (1-\omega)\sum_{j=0}^{n} \boldsymbol{V}_j - \sum_{i=0}^{n} \boldsymbol{S}_i\Big)\Big)Adj[\boldsymbol{I}_m - z\boldsymbol{S}(z^{-1})]\varphi(z)$. To evaluate the vector in the numerator of equation (17), we show that the equation $det[\boldsymbol{I}_m z - \boldsymbol{S}(z)] = 0$ has exactly m roots inside the unit circle $|z| = 1$, see Chaudhry et al. [7]. Let these roots be γ_i $(1 \leq i \leq m)$. Now, consider the zeros of the function $d(z)$. Since the equation $det[\boldsymbol{I}_m z - \boldsymbol{S}(z)] = 0$ has m roots γ_i inside the unit circle, the function $det[\boldsymbol{I}_m - z\boldsymbol{S}(z^{-1})]$ has m zeros $1/\gamma_i$ outside the unit circle $|z| = 1$. As $\pi^{-*}_{j,1}(z)$ is an analytic function of z for $|z| \le 1$, applying the partial-fraction method, we obtain

$$\pi^{-*}_{j,1}(z) = \sum_{i=1}^{m} \frac{k_{ij}}{1-\gamma_i z},\ 1 \le j \le m, \tag{19}$$

where k_{ij} are constants to be determined. Now, collecting the coefficient of z^n from both sides of (19), we have

$$\pi^{-}_{j,1}(n) = \sum_{i=1}^{m} k_{ij}\gamma_i^n,\ 1 \le j \le m,\ n \ge 0. \tag{20}$$

Now we assume $\boldsymbol{\pi}^{-}_{0}(0)$ as $\boldsymbol{\pi}^{-}_{0}(0) = \left[b_1, b_2, \ldots, b_m\right]$, where $b_1, b_2, \ldots, b_m$ are m arbitrary positive constants to be computed as described below. Hereafter, we substitute $\boldsymbol{\pi}^{-}_{0}(0)$ from above assumption into the Equation (14) and obtain

$$\boldsymbol{\pi}^{-}_{0}(n) = \boldsymbol{\pi}^{-}_{0}(0)\omega^n \mathbf{I}_m = \left[b_1, b_2, \ldots, b_m\right]\omega^n \mathbf{I}_m, \quad n \ge 0. \tag{21}$$

After this we find the constants k_{ij}'s $(1 \le i \le m,\ 1 \le j \le m)$ and b_i $(1 \le i \le m)$ by solving $m(m+1)$ linear simultaneous equations along with the following normalizing condition:

$$\sum_{j=1}^{m} \pi^{-*}_{j,1}(1) + \sum_{n=0}^{\infty}\sum_{j=1}^{m} \pi^{-}_{j,0}(n) = \sum_{j=1}^{m}\sum_{i=1}^{m} \frac{k_{ij}}{1-\gamma_i} + \sum_{j=1}^{m} \frac{b_j}{1-\omega} = 1. \tag{22}$$

The set of simultaneous linear equations are obtained from equating the corresponding components of the vector Equations (13), (15), and (16) after substituting $\boldsymbol{\pi}^{-}_{1}(n)$ and $\boldsymbol{\pi}^{-}_{0}(n)$ from Equations (20) and (21), respectively.

§5. Numerical results and discussion

To demonstrate the applicability of the results obtained in the previous sections, some numerical results have been presented in two self explanatory tables. At the bottom of the tables, several performance measures are given.

We have carried out extensive numerical work based on the procedure discussed in this paper by considering different service matrices $MSP\ (\boldsymbol{L}_0, \boldsymbol{L}_1)$ and phase-type inter-arrival time distribution $PH(\boldsymbol{\alpha}, \boldsymbol{T})$. All the calculations were performed on a PC having Intel(R) Core 2 Duo processor @1.65 GHz with 8 GB DDR2 RAM using MSPLE 18. Further, though all the numerical results were carried out in high precision, they are reported here in 6 decimal places due to lack of space.

In Table 1, we have presented various epoch probabilities for a $PH/MSP/1/\infty$ queue with exponential single vacation using our method described in this paper. Vacation time is following exponential distribution with average number of vacations per unit of time is $\gamma = 1.8$. Inter-arrival time is PH-type and its representation is given by

Pre-arrival $\pi^-_{j,0}(n)$			& $\pi^-_{j,1}(n)$		
$\pi^-_{j,0}(n)$	$j=1$	$j=2$	$j=3$	$j=4$	$\sum_{j=1}^{m=4}$
0	0.394155	0.001935	0.007826	0.001078	0.404994
1	0.012922	0.000063	0.000256	0.000035	0.013277
2	0.000424	0.000002	0.000008	0.000001	0.000435
3	0.000014	0.000000	0.000000	0.000000	0.000014
4	0.000000	0.000000	0.000000	0.000000	0.000000
5	0.000000	0.000000	0.000000	0.000000	0.000000
⋮	⋮	⋮	⋮	⋮	⋮
sum	0.407514	0.002001	0.008091	0.001114	0.418720
$\pi^-_{j,1}(n)$	$j=1$	$j=2$	$j=3$	$j=4$	$\sum_{j=1}^{m=4}$
0	0.418304	0.003215	0.006170	0.000057	0.037466
1	0.013560	0.007230	0.009807	0.006868	0.037466
2	0.001028	0.007186	0.007215	0.006678	0.037466
3	0.000530	0.006045	0.005157	0.003092	0.037466
4	0.000730	0.005106	0.004122	0.001808	0.037466
5	0.000721	0.004344	0.003459	0.001355	0.037466
⋮	⋮	⋮	⋮	⋮	⋮
sum	0.439280	0.058473	0.056044	0.027481	0.581279

Table 1: System-length distributions at a pre-arrival epoch.

$$\boldsymbol{\alpha} = \begin{bmatrix}0.22 & 0.33 & 0.45\end{bmatrix}, \boldsymbol{T} = \begin{bmatrix}-2.823 & 0.0 & 2.812\\ 3.542 & -2.942 & 1.000\\ 1.710 & 0.0 & -2.240\end{bmatrix} \text{ with } \lambda = 0.259558.$$

The MSP matrices as

$$\boldsymbol{L}_0 = \begin{bmatrix}-3.69939 & 0.01276 & 0.00572 & 0.0\\ 0.01012 & -0.55759 & 0.0 & 0.00682\\ 0.0 & 0.02343 & -0.53152 & 0.48730\\ 0.00649 & 0.55363 & 0.0 & -0.58531\end{bmatrix},$$

$$\boldsymbol{L}_1 = \begin{bmatrix}3.65748 & 0.01727 & 0.0 & 0.00616\\ 0.01353 & 0.00517 & 0.52195 & 0.0\\ 0.00924 & 0.0 & 0.0 & 0.01155\\ 0.00561 & 0.0 & 0.00847 & 0.01111\end{bmatrix},$$

with stationary mean service rate $\mu^* = 1.121972$, lag-1 correlation coefficient 0.618173 between successive service times and $\overline{\boldsymbol{\pi}} = \begin{bmatrix}0.264645 & 0.253046 & 0.254961 & 0.227348\end{bmatrix}$

so that $\rho = \lambda/(\mu^*) = 0.231341$.

Acknowledgements

The authors thank the referee for useful remarks. The second author was supported partially by NSERC under research grant number RGPIN-2014-06604.

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