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ON THE REGULARITY OF THE Q-TENSOR DEPENDING ON THE DATA

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Abstract. The coupled Navier-Stokes and Q-Tensor system is one of the models used to describe the behavior of the nematic liquid crystals, an intermediate phase between crystalline solids and isotropic fluids. These equations model the dynamics of the fluid via velocity and pressure $(\boldsymbol{u}, p)$ and the orientation of the molecules via a tensor Q. A review on the existence of weak solutions, maximum principle and a uniqueness criteria can be seen in [4] (the corresponding Cauchy problem in the whole $\mathbb{R}^3$ is analyzed by Zarnescu (cf. [10, 9])). However, the regularity of such solutions is only analyzed under some restrictive conditions: large viscosity or periodic boundary conditions.

In this work, we study two different types of regularity for the Q-Tensor model: one inherited from the usual strong solution for the Navier-Stokes equations, and another one where $(\boldsymbol{u}, Q)$ and $(\partial_t \boldsymbol{u}, \partial_t Q)$ have weak regularity (weak-t). This latter regularity is introduced due to the impossibility of obtaining local in time strong estimates for non-periodic boundary conditions, where only the existence (and uniqueness) of local weak-t solution is obtained.

Some regularity criteria for $(\boldsymbol{u}, Q)$ will also be given. In the particular case of Neumann boundary conditions for Q, the regularity criteria only must be imposed for the velocity $\boldsymbol{u}$ (cf. [5, 6]).

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§1. The model

Liquid crystals are intermediate phases of matter with properties from both solid and liquid states. The macroscopic properties come from the liquid behavior and are modeled by using the velocity and pressure $(\boldsymbol{u}, p)$. The microscopic structure enters into the model through the molecules of liquid crystals. These molecules influence the behavior of the matter and such an influence can be modeled by using different kinds of unknowns, which depend on different theories and the type of liquid crystal.

In the nematic case, where the molecules are arranged by layers and every molecule is oriented equally, two main theories appear: the Oseen-Frank theory, where the microscopic structure is modeled by using a director vector $\boldsymbol{d}$, which is the average of the main orientation of the rod-like molecules of liquid crystals, and the Landau-De Gennes theory, where the vector $\boldsymbol{d}$ is replaced by a tensor Q.

The tensor Q is related to the second moment of a probability measure $\mu(\boldsymbol{x}, \cdot) : \mathcal{L}(\mathbb{S}^2) \to [0, 1]$ for each $\boldsymbol{x} \in \Omega$, being $\mathcal{L}(\mathbb{S}^2)$ the family of Lebesgue measurable sets on the unit sphere. For any $A \subset \mathbb{S}^2$, $\mu(\boldsymbol{x}, A)$ is the probability that the molecules with centre of mass in a very small neighborhood of the point $\boldsymbol{x} \in \Omega$ are pointing in a direction contained in A. From a physical point of view, this probability (cf. [3, 8]) must satisfy $\mu(\boldsymbol{x}, A) = \mu(\boldsymbol{x}, -A)$ in order to reproduce the so-called "head-to-tail" symmetry. As a consequence, the first moment of the probability measure vanishes, that is

$$\langle p\rangle(\boldsymbol{x}) = \int_{\mathbb{S}^2} p_i \, d\mu(\boldsymbol{x}, p) = 0.$$

Then, the main information on μ comes from the second moment tensor

$$M(\mu)_{ij} = \int_{\mathbb{S}^2} p_i \, p_j \, d\mu(p), \quad i, j = 1, 2, 3.$$

As a consequence, $M(\mu) = M(\mu)^t$ and $tr(M) = 1$. If the orientation of the molecules is equally distributed, then the distribution is isotropic and $\mu = \mu_0$, $d\mu_0(p) = \frac{1}{4\pi} \, dA$ and $M(\mu_0) = \frac{1}{3} \, \mathbb{I}$. The deviation of the second moment tensor from its isotropic value is therefore measured as:

$$Q = M(\mu) - M(\mu_0) = \int_{\mathbb{S}^2} \left(p \otimes p - \frac{1}{3} \, \mathbb{I} \right) d\mu(p),$$

which is the definition for the tensor Q, and from which the symmetry and traceless for Q is deduced.

By following the Landau-De Gennes theory, a model to study the behavior of nematic liquid crystals filling a bounded domain $\Omega \subset \mathbb{R}^3$, with boundary $\partial\Omega$, is given by:

$$\left\{ \begin{array}{l} D_t \boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla p = \nabla \cdot \tau(Q) + \nabla \cdot \sigma(H, Q), \quad \nabla \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega \times (0, T), \\ D_t Q - S(\nabla \boldsymbol{u}, Q) = -\gamma \, H(Q) \quad \text{in } \Omega \times (0, T), \end{array} \right. \tag{1}$$

where $\boldsymbol{u} : (0, T) \times \Omega \to \mathbb{R}^3$ is the velocity field, $p : (0, T) \times \Omega \to \mathbb{R}$ is the pressure and $Q : (0, T) \times \Omega \to \mathbb{R}^{3\times 3}$ is a symmetric and traceless tensor. The operator $D_t(\cdot) = \partial_t(\cdot) + (\boldsymbol{u} \cdot \nabla)(\cdot)$ is the material derivative, $\nu > 0$ is the viscosity coefficient, and $\gamma > 0$ is a material-dependent elastic constant. The tensors $\tau = \tau(Q)$, $\sigma = \sigma(H, Q)$ are defined by:

$$\left\{ \begin{array}{rcl} \tau_{ij}(Q) & = & -\varepsilon \left(\partial_j Q : \partial_i Q \right) = -\varepsilon \, \partial_j Q_{kl} \, \partial_i Q_{kl} \quad (\varepsilon > 0), \\ \sigma(H, Q) & = & H \, Q - Q \, H, \end{array} \right. \tag{2}$$

where $H = H(Q) = -\varepsilon \, \Delta Q + f(Q)$ and

$$f(Q) = a \, Q - b \, (Q^2 - \frac{1}{3} \, tr(Q^2)) + c|Q|^2 \, Q \quad \text{with } a, b \in \mathbb{R} \text{ and } c > 0, \tag{3}$$

hereafter, $|Q|^2 = Q : Q = \sum_{i,j=1}^{3} Q_{ij} \, Q_{ij}$ denotes the tensor euclidean norm. The stretching term $S(\nabla \boldsymbol{u}, Q)$ is given by

$$S(\nabla \boldsymbol{u}, Q) = \boldsymbol{W} \, Q^t - Q^t \, \boldsymbol{W} \tag{4}$$

where $\boldsymbol{W} = \frac{1}{2} \, (\nabla \boldsymbol{u} - (\nabla \boldsymbol{u})^t)$ is the antisymmetric part of the gradient of $\boldsymbol{u}$. Definition (4) guarantees that any weak solution provides a symmetric tensor Q; meanwhile if $f(Q)$ is taken as in (3), any weak solution provides a traceless tensor Q ([7]). Both physical restrictions are then satisfied.

The PDE system is enclosed with the following initial and boundary conditions:

$$\boldsymbol{u}|_{t=0} = \boldsymbol{u}_0, \quad Q|_{t=0} = Q_0 \quad \text{in } \Omega, \tag{5}$$

$$\boldsymbol{u}|_\Gamma = \mathbf{0} \quad \text{in } (0,T), \tag{6}$$

and

$$\text{either } \partial_{\mathbf{n}} Q|_\Gamma = 0 \quad \text{or} \quad Q|_\Gamma = Q_\Gamma \quad \text{in } (0,T), \tag{7}$$

The compatibility condition $\partial_{\mathbf{n}} Q_0|_{\partial\Omega} = 0$ must be satisfied if $(7)_1$ is considered, and $Q_0|_\Gamma = Q_\Gamma(0)$ if a time-dependent boundary data Q_Γ is taken in $(7)_2$.

For the initial-value problem (1)-(5) in the whole $\Omega = \mathbb{R}^3$, the existence of global in time weak solutions in $3D$, and strong regularity and weak-strong uniqueness results in $2D$ are proved in [10]. A generalized model considering more stretching effects is studied by the same author in [9]. When the initial and boundary-value problem (1)-(7) is considered in a bounded domain $\Omega \subset \mathbb{R}^3$, we prove the existence of global weak solutions and the weak/strong uniqueness of this model (cf. [7]), and the local regularity and uniqueness (cf. [5]). Some of these last results are also proved in [2] via a different approximation.

§2. Main results

A global in time weak solutions for the problem (1)-(7) has the following regularity

$$\begin{cases} \boldsymbol{u} \in L^\infty(0,+\infty;\mathbf{L}^2(\Omega)) \cap L^2(0,+\infty;\mathbf{H}^1(\Omega)), \\ Q \in L^\infty(0,+\infty;\mathbb{H}^1(\Omega)) \cap L^2(0,T;\mathbb{H}^2(\Omega)), \quad \forall\, T > 0, \end{cases} \tag{8}$$

and satisfies a variational formulation of the $\boldsymbol{u}$-system $(1)_a$ and the Q-system $(1)_b$ pointwisely. The proof of the existence of global weak solutions [7]) is based on the energy equality:

$$\frac{d}{dt}\left(\frac{1}{2}\|\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \int_\Omega \mathcal{E}(Q)\,d\boldsymbol{x}\right) + \nu\|\nabla\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \gamma\|H\|^2_{\mathbb{L}^2(\Omega)} = 0, \tag{9}$$

where $\mathcal{E}(Q) = \dfrac{\varepsilon}{2}|\nabla Q|^2 + F(Q)$ and $F(Q)$ is a potential function defined by

$$F(Q) = \frac{a}{2}\,|Q|^2 - \frac{b}{3}\,(Q^2 : Q) + \frac{c}{4}\,|Q|^4,$$

with $a, b \in \mathbb{R}$ and $c > 0$ ($f(Q)$ is the derivative of $F(Q)$). The finite-time boundedness of ΔQ in $L^2(0,T;\mathbb{H}^2(\Omega))$ only holds for any finite T, and it is deduced from the large time regularity $H \in L^2(0,+\infty;\mathbb{L}^2(\Omega))$ and $F(Q) \in L^\infty(0,+\infty;L^1(\Omega))$, owing to (9).

Similar to the Navier-Stokes equations, the strong solution for this model can be defined as:

$$\begin{cases} \boldsymbol{u} \in L^\infty(0,T;\mathbf{H}^1(\Omega)) \cap L^2(0,T;\mathbf{H}^2(\Omega)), \quad \partial_t\boldsymbol{u} \in L^2(0,T;\mathbf{L}^2(\Omega)), \\ Q \in L^\infty(0,T;\mathbb{H}^2(\Omega)) \cap L^2(0,T;\mathbb{H}^3(\Omega)), \quad \partial_t Q \in L^\infty(0,T;\mathbb{L}^2(\Omega)) \cap L^2(0,T;\mathbb{H}^1(\Omega)). \end{cases}$$

However, the local in time existence of strong solutions is difficult to obtain when boundary conditions are not periodic. Nevertheless, in the particular case of $Q|_\Gamma = 0$, the following result can be proved (cf. [5]) (the existence of local in time strong solutions when $Q|_{\partial\Omega} = 0$ are also obtained via a fixed-point argument in [1]):

Theorem 1 (**Strong solution for** $Q|_\Gamma = 0$ **and** $S(\cdot,\cdot)$ **given in (4)).** *Let us consider the problem (1)-(7) with homogeneous-Dirichlet conditions for Q. Then, there exists a unique strong solution* $(\boldsymbol{u}, Q)$ *in* $(0, T^*)$ *where either* T^* *is small enough or* $T^* = T$ *(*$T > 0$ *fixed) whether:*

$$\nabla \boldsymbol{u} \in L^{2q/(2q-3)}(0,T;\mathbf{L}^q(\Omega)), \quad 2 \le q \le 3.$$

The control on boundary integral terms appearing in the proof of the previous result when the boundary conditions are different from $Q|_\Gamma = 0$ remains as an open problem.

In order to circumvent the difficulty of obtaining strong solutions for more general boundary conditions for Q, we analyze what we call "weak-t" solution:

$$\left\{ \begin{array}{l} \partial_t \boldsymbol{u} \in L^\infty(0,T;\mathbf{L}^2(\Omega)) \cap L^2(0,T;\mathbf{H}^1(\Omega)),\ \boldsymbol{u} \in L^\infty(0,T;\mathbf{H}^1(\Omega)), \\ \partial_t Q \in L^\infty(0,T;\mathbb{H}^1(\Omega)) \cap L^2(0,T;\mathbb{H}^2(\Omega)),\ Q \in L^\infty(0,T;\mathbb{H}^2(\Omega)), \end{array} \right. \tag{10}$$

which is a weak solution $(\boldsymbol{u}, Q)$ where $(\partial_t \boldsymbol{u}, \partial_t Q)$ also have the weak regularity such as (8). Thus, the existence and uniqueness of weak-t solutions can be obtained (cf. [5]). But, the concept of weak-t regularity is different from the strong one. Note that weak-t regularity (10) does not implies $\boldsymbol{u} \in L^2(0,T;\mathbf{H}^2(\Omega))$ and $Q \in L^2(0,T;\mathbb{H}^3(\Omega))$.

The main results about the existence of a local in time weak-t solution for the Q-tensor model are summarized below (cf. [5]):

Theorem 2 (**Local in time weak-t regularity for time-independent b.c.).** *Let* $(\boldsymbol{u}, Q)$ *be a weak solution in* $(0,T)$ *of the (QT) problem (1)-(7) (in the Dirichlet case,* Q_Γ *is independent on the time). Assume* $(\boldsymbol{u}_0, Q_0) \in \mathbf{H}^2(\Omega) \times \mathbb{H}^3(\Omega)$*. Then, there exists a time* $T^*(\le T)$ *such that* $(\boldsymbol{u}, Q)$ *is the unique weak-t solution in* $(0, T^*)$*, i.e. satisfying the weak-t regularity (10) for* $T = T^*$*.*

A similar result appears in [2] where the authors have analyzed the weak-t regularity for a more complete Q-tensor model appearing in [9] with Neumann boundary conditions for Q, but the argument made in [2] is different from ours.

The case of time-dependent Dirichlet data for Q, that is $Q|_\Gamma = Q_\Gamma$ with $Q_\Gamma = Q_\Gamma(t)$, uses the following lifting function $\widetilde{Q}$:

$$\partial_t \widetilde{Q} - \gamma\,\varepsilon\,\Delta \widetilde{Q} = 0 \quad \text{in } (0,T)\times\Omega, \quad \widetilde{Q}|_\Gamma = Q_\Gamma, \quad \widetilde{Q}|_{t=0} = Q_0.$$

Theorem 3 (**Local in time weak-t solution for time-dependent b.c.).** *Let* $(\boldsymbol{u}, Q)$ *be a weak solution in* $(0,T)$ *of the (QT) problem (1)-(6), with time-dependent Dirichlet boundary condition* $(7)_2$ *for Q. Assume* $(\boldsymbol{u}_0, Q_0) \in \mathbf{H}^2(\Omega) \times \mathbb{H}^3(\Omega)$ *and* $\partial_t \widetilde{Q} \in L^\infty(0,T;\mathbb{H}^1(\Omega)) \cap L^4(0,T;\mathbb{H}^2(\Omega))$*. Then there exists a time* T^* *such that* $(\boldsymbol{u}, Q)$ *is the unique weak-t solution in* $(0, T^*)$*.*

Moreover, using additional regularity for $\nabla \boldsymbol{u}$ and ΔQ instead of smallness of the time T^* it is possible to prove:

Theorem 4 (Regularity criteria for global in time weak-t regularity). *Let* $(\boldsymbol{u}, Q)$ *be a weak solution in* $(0,T)$ *of the problem (1)–(7), having the additional regularity:*

$$\begin{cases} \nabla \boldsymbol{u} \in L^{2q/(2q-3)}(0,T;\mathbf{L}^q(\Omega)), & 3/2 \le q \le 3, \\ \Delta Q \in L^{2s/(2s-3)}(0,T;\mathbb{L}^s(\Omega)), & 3/2 \le s \le 3. \end{cases} \tag{11}$$

Assume $(\boldsymbol{u}_0, Q_0) \in \mathbf{H}^2(\Omega)\times\mathbb{H}^3(\Omega)$ *and for the case of time-dependent Dirichlet boundary conditions,* $\partial_t \widetilde{Q} \in L^\infty(0,T;\mathbb{H}^1(\Omega)) \cap L^4(0,T;\mathbb{H}^2(\Omega))$*. Then,* $(\boldsymbol{u}, Q)$ *is the unique weak-t solution of the system in the whole time interval* $(0,T)$*.*

In the particular case that homogeneous Neumann boundary conditions for Q given in $(7)_1$ are considered, the regularity hypothesis $(11)_2$ for Q can be removed:

Theorem 5. *Let* $(\boldsymbol{u}, Q)$ *be a weak solution in* $(0,T)$ *of (1)-(6) and* $(7)_1$*. If* $(\boldsymbol{u}_0, Q_0) \in \mathbf{H}^2(\Omega)\times \mathbb{H}^3(\Omega)$ *and* $\nabla\boldsymbol{u}$ *has the additional regularity:*

$$\nabla \boldsymbol{u} \in L^{5/2}(0,T;\mathbf{L}^{5/2}(\Omega)), \tag{12}$$

then $(\boldsymbol{u}, Q)$ *is the unique weak-t solution in* $(0,T)$*.*

Moreover, hypothesis (12) guarantees the uniqueness of weak solutions because it removes the hypothesis for ΔQ appearing in Theorem 2 of [7]:

Corollary 6 (Uniqueness criteria). *Assume* $(\boldsymbol{u}_0, Q_0) \in \mathbf{L}^2(\Omega)\times \mathbb{W}^{2-2/(5/2),5/2}(\Omega)$*. Let* $(\boldsymbol{u}, Q)$ *be a weak solution of (1)-(6) and* $(7)_1$ *such that* $\nabla\boldsymbol{u}$ *satisfies the regularity criterion (12). Then, this solution coincides in* $(0,T)$ *with any weak solution associated to the same data.*

In this work we only point out the main steps and difficulties appearing into the proof of the previous results. Concretely, in Section 3 we will treat the proof of Theorem 1 focusing in the terms that only work well for homogeneous Dirichlet boundary conditions for Q. In Section 4, a sketch of the proof of Theorem 2 will be done. Finally, in Section 5 we will briefly give the main steps and results used to prove Theorem 5.

§3. Sketch of the proof of Theorem 1

We multiply $\boldsymbol{u}$-system $(1)_1$ by $A\boldsymbol{u} := P_H(-\Delta\boldsymbol{u})$ and Q-system $(1)_2$ by $-\Delta H$, obtaining:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla\boldsymbol{u}\|^2_{L^2(\Omega)} + \nu\,\|A\boldsymbol{u}\|^2_{L^2(\Omega)} &= -((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, A\boldsymbol{u}) + ((A\boldsymbol{u}\cdot\nabla)Q, H) \\ &- (\sigma(H,Q), \nabla(A\boldsymbol{u})) := K_1 + K_2 + K_3 \end{aligned} \tag{13}$$

and

$$(\nabla(\partial_t Q), \nabla H) + \gamma\|\nabla H\|^2_{L^2(\Omega)} = -(\nabla(\boldsymbol{u}\cdot\nabla Q), \nabla H) + (\nabla S(\nabla\boldsymbol{u}, Q), \nabla H) := K_4 + K_5. \tag{14}$$

We want to bound (13) and (14) jointly. Firstly, we rewrite the term $(\nabla\partial_t Q, \nabla H)$ as:

$$\begin{aligned} (\nabla\partial_t Q, \nabla H) &= (\partial_t(-\Delta Q), H) + \int_\Gamma \partial_{\mathbf{n}}(\partial_t Q)\, H\, d\sigma \\ &= \frac{1}{2\varepsilon}\frac{d}{dt}\|H\|^2_{\mathbb{L}^2(\Omega)} - \frac{1}{\varepsilon}(\partial_t(f(Q)), H) := \frac{1}{2\varepsilon}\frac{d}{dt}\|H\|^2_{\mathbb{L}^2(\Omega)} - K_0, \end{aligned}$$

where the boundary term vanishes if $H(Q)|_\Gamma = 0$.

As $K_3 + K_5 = (\nabla S(\nabla \boldsymbol{u}, Q), \nabla H) - (\sigma(H, Q), \nabla(A\boldsymbol{u}))$, using that

$$S(\nabla(\Delta \boldsymbol{u}), Q) : H = \sigma(H, Q) : \nabla(\Delta \boldsymbol{u}), \tag{15}$$

and the integration by parts

$$(\nabla S(\nabla \boldsymbol{u}, Q), \nabla H) = -(\Delta S(\nabla \boldsymbol{u}, Q), H) + \int_\Gamma \partial_{\mathbf{n}} S(\nabla \boldsymbol{u}, Q)\, H\, d\sigma = -(\Delta S(\nabla \boldsymbol{u}, Q), H)$$

again if $H(Q)|_\Gamma = 0$. Since $S(\cdot,\cdot)$ is quadratic,

$$\Delta S(\nabla \boldsymbol{u}, Q) = S(\nabla(\Delta \boldsymbol{u}), Q) + 2S(\nabla(\nabla \boldsymbol{u}), \nabla Q) + S(\nabla \boldsymbol{u}, \Delta Q),$$

hence using (15),

$$K_3 + K_5 = -(\sigma(H, Q), \nabla(\Delta \boldsymbol{u} + A\boldsymbol{u})) - 2(S(\nabla(\nabla \boldsymbol{u}), \nabla Q), H) - (S(\nabla \boldsymbol{u}, \Delta Q), H).$$

The worse term to manage with is $-(\sigma(H, Q), \nabla(\Delta \boldsymbol{u} + A\boldsymbol{u}))$ because it cannot be controlled directly with the left hand side of (13). Using the Helmholtz decomposition $-\Delta \boldsymbol{u} = A\boldsymbol{u} + \nabla\pi$, with $\int_\Omega \pi\, d\boldsymbol{x} = 0$, and

$$-(\sigma(H, Q), \nabla(\Delta \boldsymbol{u} + A\boldsymbol{u})) = (\sigma(H, Q), \nabla(\nabla\pi)) = (\sigma_s(H, Q), \nabla(\nabla\pi)), \tag{16}$$

where $\sigma_s = (\sigma + \sigma^t)/2$ denotes the symmetric part of σ.

In summary, the strong solution can be assured if $H(Q)|_\Gamma$ and (16) vanish. The later holds when $\sigma(H, Q)$ is antisymmetric, which is for instance the case when Q is symmetric. The Q-tensor model having the stretching term defined as (4) guarantees the symmetry of Q (cf. [7]). The vanishing boundary condition $H(Q)|_\Gamma = 0$ holds if $Q|_\Gamma = 0$ but not for other boundary conditions in (7).

§4. A sketch of the proof of Theorem 2

First of all, taking $\gamma \partial_t H$ as test function in the Q-system $(1)_2$,

$$\gamma(\partial_t Q, \partial_t H) + \gamma((\boldsymbol{u}\cdot\nabla)Q, \partial_t H) - \gamma(S(\nabla \boldsymbol{u}, Q), \partial_t H) + \frac{\gamma^2}{2}\frac{d}{dt}\|H\|^2_{\mathbb{L}^2(\Omega)} = 0,$$

and bounding in adequate Sobolev spaces, we obtain:

$$\begin{aligned} \gamma\,\varepsilon\,\|\partial_t(\nabla Q)\|^2_{\mathbb{L}^2(\Omega)} &+ \frac{\gamma^2}{2}\frac{d}{dt}\|H\|^2_{\mathbb{L}^2(\Omega)} \le \gamma\, C\left(1 + \|Q\|_{\mathbb{H}^2(\Omega)}\right)\|\partial_t Q\|^2_{\mathbb{L}^2(\Omega)} \\ &+ \delta\gamma\,\|\partial_t H\|^2_{\mathbb{L}^2(\Omega)} + \gamma\, C_\delta\,\|Q\|_{\mathbb{H}^2(\Omega)}\,\|\nabla \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)}. \end{aligned} \tag{17}$$

Second, taking $\partial_t Q$ as test function in $\partial_t(1)_2$, one has

$$\frac{1}{2}\frac{d}{dt}\|\partial_t Q\|^2_{\mathbb{L}^2(\Omega)} + (\partial_t \boldsymbol{u}\cdot\nabla Q, \partial_t Q) - (S(\nabla(\partial_t \boldsymbol{u}), Q), \partial_t Q) + \gamma(\partial_t H, \partial_t Q) = 0, \tag{18}$$

where we have used that $(\boldsymbol{u}\cdot\nabla(\partial_t Q),\partial_t Q)=0$ and $(S(\nabla\mathbf{u},\partial_t Q),\partial_t Q)=0$. Bounding each term, (18) becomes:

$$\begin{aligned}\frac{d}{dt}\|\partial_t Q\|^2_{\mathbb{L}^2(\Omega)} &+ \gamma\,\varepsilon\,\|\nabla(\partial_t Q)\|^2_{\mathbb{L}^2(\Omega)} \\ &\leq C_{\gamma,\varepsilon}\left(1+\|Q\|_{\mathbb{H}^2(\Omega)}\right)\left(\|\partial_t\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)}+\|\partial_t Q\|^2_{\mathbb{L}^2(\Omega)}\right).\end{aligned}\tag{19}$$

Adding (17) to (19), we obtain:

$$\begin{aligned}\frac{d}{dt}\Big(\|\partial_t Q\|^2_{\mathbb{L}^2(\Omega)} &+ \frac{\gamma^2}{2}\,\|H\|^2_{\mathbb{L}^2(\Omega)}\Big)+\gamma\,\varepsilon\,\|\partial_t(\nabla Q)\|^2_{\mathbb{L}^2(\Omega)} \\ &\leq C_{\gamma,\varepsilon}\left(1+\|Q\|_{\mathbb{H}^2(\Omega)}\right)\left(\|\partial_t\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)}+\|\partial_t Q\|^2_{\mathbb{L}^2(\Omega)}\right) \\ &+ \gamma\,\delta\,\|\partial_t H\|^2_{\mathbb{L}^2(\Omega)}+\gamma\,C_\delta\,\|Q\|_{\mathbb{H}^2(\Omega)}\,\|\nabla\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)}.\end{aligned}\tag{20}$$

Taking $\partial_t\boldsymbol{u}$ as test function in $\boldsymbol{u}$-system $(1)_1$, we obtain:

$$\|\partial_t\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)}+\frac{\nu}{2}\frac{d}{dt}\|\nabla\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)}=-((\boldsymbol{u}\cdot\nabla)\boldsymbol{u},\partial_t\boldsymbol{u})+\int_\Omega(\partial_t\boldsymbol{u}\cdot\nabla)Q:H\,d\boldsymbol{x}+(\sigma(H,Q),\nabla(\partial_t\boldsymbol{u})),$$

arriving at the inequality:

$$\begin{aligned}\frac{\nu}{2}\frac{d}{dt}\|\nabla\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} &+ \|\partial_t\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)}\leq\delta\,\|\nabla(\partial_t\boldsymbol{u})\|^2_{\mathbf{L}^2(\Omega)} \\ &+ C_\delta\left(\|\nabla\boldsymbol{u}\|^3_{\mathbf{L}^2(\Omega)}+\|Q\|_{\mathbb{H}^2(\Omega)}\|H\|^2_{\mathbb{L}^2(\Omega)}\right).\end{aligned}\tag{21}$$

Taking $\partial_t\boldsymbol{u}$ as test function in $\partial_t(1)_1$, $-\varepsilon\,\Delta(\partial_t Q)$ as test function in $\partial_t(1)_2$, and assuming that the boundary term $\varepsilon\displaystyle\int_\Gamma\partial^2_{tt}Q:\partial_{\mathbf{n}}(\partial_t Q)\,d\sigma$ vanishes (which is true for either homogeneous Neumann or time-independent Dirichlet boundary conditions), we obtain:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\partial_t\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)}+\varepsilon\,\|\nabla(\partial_t Q)\|^2_{\mathbb{L}^2(\Omega)}\right)+\nu\|\nabla(\partial_t\mathbf{u})\|^2_{\mathbf{L}^2(\Omega)}+\gamma\,\varepsilon^2\,\|\Delta(\partial_t Q)\|^2_{\mathbb{L}^2(\Omega)} \\ &=-(\partial_t\boldsymbol{u}\cdot\nabla\boldsymbol{u},\partial_t\boldsymbol{u})+(\partial_t\mathbf{u}\cdot\nabla Q,\partial_t(f(Q)))+(\partial_t\boldsymbol{u}\cdot\nabla(\partial_t Q),H) \\ &-(\sigma\left(\partial_t(f(Q)),Q\right),\nabla(\partial_t\boldsymbol{u}))-(\sigma(H,\partial_t Q),\nabla(\partial_t\boldsymbol{u}))-(\boldsymbol{u}\cdot\nabla(\partial_t Q),-\varepsilon\,\Delta(\partial_t Q)) \\ &+(S(\nabla\boldsymbol{u},\partial_t Q),-\varepsilon\Delta(\partial_t Q))-\gamma\,(\partial_t(f(Q)),-\varepsilon\Delta(\partial_t Q)):=\sum_{i=1}^{8}I_i.\end{aligned}\tag{22}$$

Therefore adding ε(19) and (22),

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\partial_t\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)}+\varepsilon\,\|\partial_t Q\|^2_{\mathbb{H}^1(\Omega)}\right)+C_1\,\nu\|\partial_t\boldsymbol{u}\|^2_{\mathbf{H}^1(\Omega)}+C_2\,\gamma\,\varepsilon^2\,\|\partial_t Q\|^2_{\mathbb{H}^2(\Omega)} \\ &\leq C_{\gamma,\varepsilon}\left(1+\|Q\|_{\mathbb{H}^2(\Omega)}\right)\left(\|\partial_t\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)}+\|\partial_t Q\|^2_{\mathbb{L}^2(\Omega)}\right)+\sum_{i=1}^{8}I_i.\end{aligned}$$

Every I_i-term will be bounded looking for a local in time weak-t estimate. Concretely:

$$\begin{aligned}&\frac{d}{dt}\left(\|\partial_t \boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \varepsilon\,\|\partial_t Q\|^2_{\mathbb{H}^1(\Omega)}\right) + C_1\,\nu\|\partial_t \boldsymbol{u}\|^2_{\mathbf{H}^1(\Omega)} + C_2\,\gamma\varepsilon^2\,\|\partial_t Q\|^2_{\mathbb{H}^2(\Omega)}\\ &\le a(t)\|\partial_t Q\|^2_{\mathbb{H}^1(\Omega)} + C_{\gamma,\nu,\varepsilon}\left(\|\nabla\boldsymbol{u}\|^4_{\mathbf{L}^2(\Omega)} + \|H\|^4_{\mathbb{L}^2(\Omega)}\right)\left(\|\partial_t\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \|\partial_t Q\|^2_{\mathbb{H}^1(\Omega)}\right),\end{aligned} \tag{23}$$

where $a \in L^1(0,T)$ is defined as:

$$a(t) = C_{\gamma,\varepsilon,\nu}\left(1 + \|Q\|_{\mathbb{H}^2(\Omega)}\right)$$

Observe that from (23) we cannot obtain any estimate in time yet. But, adding (23) to the estimate for Q given by (20) and the estimate for $\boldsymbol{u}$ given by (21), we arrive to:

$$\begin{aligned}&\frac{d}{dt}\left(\|\partial_t\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \varepsilon\,\|\partial_t Q\|^2_{\mathbb{H}^1(\Omega)} + \gamma^2\,\varepsilon\,\|H\|^2_{\mathbb{L}^2(\Omega)} + \nu\,\|\nabla\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)}\right)\\ &+C_1\,\frac{\nu}{2}\|\partial_t\boldsymbol{u}\|^2_{\mathbf{H}^1(\Omega)} + C_2\,\frac{\gamma\varepsilon^2}{2}\,\|\partial_t Q\|^2_{\mathbb{H}^2(\Omega)}\\ &\le \widetilde{a}(t)\left(\|\partial_t\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \varepsilon\,\|\partial_t Q\|^2_{\mathbb{H}^1(\Omega)} + \gamma^2\,\varepsilon\,\|H\|^2_{\mathbb{L}^2(\Omega)} + \nu\,\|\nabla\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)}\right)\\ &+C_{\nu,\gamma,\varepsilon}\left(\|H\|^4_{\mathbb{L}^2(\Omega)} + \|\nabla\boldsymbol{u}\|^4_{\mathbf{L}^2(\Omega)}\right)\left(\|\partial_t\boldsymbol{u}\|^2_{\mathbf{L}^2(\Omega)} + \varepsilon\,\|\partial_t Q\|^2_{\mathbb{H}^1(\Omega)}\right),\end{aligned}$$

where

$$\widetilde{a}(t) = a(t) + C_{\gamma,\varepsilon}\,\|\nabla\boldsymbol{u}\|_{\mathbf{L}^2(\Omega)} \equiv C_{\gamma,\varepsilon,\nu}\left(1 + \|Q\|^2_{\mathbb{H}^2(\Omega)} + \|\nabla\boldsymbol{u}\|_{\mathbf{L}^2(\Omega)}\right).$$

Integrating in time, and choosing regular enough initial data, we obtain the existence of a small enough time T^* such that (10) hold for $T = T^*$.

§5. The case of $\partial_{\mathbf{n}} Q|_\Gamma = 0$ (sketch of the proof of Theorem 5)

The key point are the uniqueness criteria for weak solutions (cf. Theorem 2 in [7]) and the two following results (see [6] for the complete proof):

Theorem 7. *Let $(\boldsymbol{u}, Q)$ be a weak solution of (1)-(6) and $(7)_1$ such that $\nabla\boldsymbol{u}$ satisfies $(11)_1$ and Q holds the maximum principle:*

$$Q \in L^\infty(0,T;\mathbb{L}^\infty(\Omega)). \tag{24}$$

Then, $\nabla Q \in L^\infty(0,T;\mathbb{L}^3(\Omega)) \cap L^3(0,T;\mathbb{L}^9(\Omega))$.

Theorem 8. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega$ of class $C^{2+\epsilon}$ for some $\epsilon > 0$. Let $(\boldsymbol{u}, Q)$ be a weak solution of (1)-(6) and $(7)_1$ with Q satisfying (24). Assume $(\boldsymbol{u}_0, Q_0) \in \mathbf{L}^2(\Omega)\times\mathbb{B}^{2-2/\gamma}_\gamma(\Omega)$ and hypotesis $(11)_1$ for $\nabla\boldsymbol{u}$, where $\gamma = \min\{q, 2q/(2q-3)\}$ and $3/2 \le q \le 3$ is the exponent given in $(11)_1$. Moreover, under the following hypothesis for the tensor:*

$$\nabla Q \in L^\infty(0,T;\mathbb{L}^3(\Omega)),$$

the solution $(\boldsymbol{u}, Q)$ satisfies the additional regularity

$$Q \in L^\gamma(0,T;\mathbf{W}^{2,\gamma}(\Omega)) \quad \textit{and} \quad \partial_t Q \in L^\gamma(0,T;\mathbf{L}^\gamma(\Omega)).$$

In this last result, we use Theorem 17 of a Solonnikov's paper (cf. [11]) for the case of Neumann boundary conditions.

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References

[1] Abels, H., Dolzmann, G., and Liu, Y. Strong solutions for the beris-edwards model for nematic liquid crystals with homogeneous dirichlet boundary conditions. *arXiv:1312.5988*. Available from: `http://arxiv.org/abs/1312.5988`.

[2] Abels, H., Dolzmann, G., and Liu, Y. Well-posedness of a fully coupled Navier-Stokes/Q-tensor system with inhomogeneous boundary data. *SIAM J. Math. Anal. 46* (2014), 3050–3077.

[3] de Gennes, P., and Prost, J. *The Physics of Liquid Crystals*. International Series of Monographs on Physics. Clarendon Press, 1995.

[4] Guillén-González, F., and Rodríguez-Bellido, M. Á. Some properties on the Q-tensor system. *Monogr. Mat. García Galdeano 39* (2014), 133–145.

[5] Guillén-González, F., and Rodríguez-Bellido, M. Á. Weak time regularity and uniqueness for a Q-tensor model. *SIAM J. Math. Anal. 46* (2014), 3540–3567.

[6] Guillén-González, F., and Rodríguez-Bellido, M. Á. A uniqueness and regularity criterium for q-tensor models with neumann boundary conditions. *Differential Integral Equations, 28*, 5-6 (2015), 537–552.

[7] Guillén-González, F., and Rodríguez-Bellido, M. Á. Weak solutions for an initial-boundary Q-tensor problem related to liquid crystals. *Nonlinear Anal. 112* (2015), 84–104.

[8] Majumdar, A., and Zarnescu, A. Landau-De Gennes theory of nematic liquid crystals: the Oseen-Frank limit and beyond. *Arch. Ration. Mech. Anal. 196* (2010), 227–280.

[9] Paicu, M., and Zarnescu, A. Global existence and regularity for the full coupled Navier-Stokes and Q-tensor system. *SIAM Journal on Mathematical Analysis 43* (2011), 2009–2049.

[10] Paicu, M., and Zarnescu, A. Energy dissipation and regularity for a coupled Navier-Stokes and Q-tensor system. *Arch. Ration. Mech. Anal. 203*, 1 (2012), 45–67.

[11] Solonnikov, V. A. A priori estimates for solutions of second-order equations of parabolic type. *Trudy Mat. Inst. Steklov. 70* (1964), 133–212 (English transl., Amer. Math. Soc. Transl. (2) **65** (1967), 51–137).

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