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# American options and nonlinear Black-Scholes equations - A viscosity solution approach

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**Abstract.** This article studies the existence and uniqueness of the American option pricing problem for nonlinear Black-Scholes equations. The used method is based on the viscosity solution approach.

*Keywords:* Nonlinear Black-Scholes equations, viscosity solutions.

*AMS classification:* 35K15, 35K67, 91G80.

## §1. Introduction

The usual Black-Scholes-Merton model [6], developed in 1973, represents the foundation of modern option pricing theory. In the following years there have been several approaches to generalize this model, for example by introducing transaction costs (cf. [4], [7], [12], [16]), influence of large traders (cf. [9], [10], [13], [17], [18], [19]), volatility uncertainty (cf. [3]), jump diffusion (cf. [8], [14], [15], [22]) or fractional Brownian motions (cf. [20]). These approaches have in common that the resulting PDE is fully nonlinear, which means that the considered problem has the form

$$\begin{cases} V_t(t,S) + \mathcal{BS}(t,S,V_S,V_{SS})\,V(t,S) = 0 & (t,S) \in [0,T) \times (0,\infty) \\ V(T,S) = g(S) & S \in (0,\infty), \end{cases} \tag{1.1}$$

where $\mathcal{BS}$ is the nonlinear Black-Scholes operator which is given by

$$\mathcal{BS}(t,S,V_S,V_{SS}) \overset{\text{def}}{=} \frac{\tilde{\sigma}^2(t,S,V_S,V_{SS})}{2} S^2 \frac{\partial^2 \cdot}{\partial S^2} + (r(t) - q(t,S)) S \frac{\partial \cdot}{\partial S} - r(t) \cdot + \mathcal{I}[\cdot],$$

$V : [0,T] \times (0,\infty) \to \mathbb{R}$ is the pricing function, $g : (0,\infty) \to \mathbb{R}$ the payoff function, $r : [0,T] \to \mathbb{R}$ the risk-free interest rate, $q : [0,T] \times (0,\infty) \to \mathbb{R}$ the continuously paid dividend, $\mathcal{I}$ is a nonlocal integral term and $\tilde{\sigma} : [0,T] \times (0,\infty) \times \mathbb{R} \times \mathbb{R} \to [0,\infty)$ the modified volatility function which depends on the specific model.

Furthermore, there have been several approaches to handle the American option problem. The latest approach by Benth et al. (cf. [5]) works with the formulation of the problem as viscosity solution.

**Definition 1.** Consider a general partial differential equation of the following form

$$\begin{cases} u_t(t,x) + F(t,x,u(t,x),\mathcal{I}u(t,x),D_x u(t,x),D_{xx}u(t,x)) = 0 & (t,x) \in (0,T] \times \mathbb{R}^N \\ u(0,x) = g(x) & x \in \mathbb{R}^N, \end{cases} \tag{1.2}$$

for some (nonlinear) function $F$ defined on $\Gamma \stackrel{\text{def}}{=} [0,T]\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{S}^N$, where $\mathbb{S}^N$ denotes the space of $N\times N$ symmetric matrices with the usual ordering. (We say that $A \le B$ if $\langle Ax, x\rangle \le \langle Bx, x\rangle$ for all $x \in \mathbb{R}^N$ with $|x| \le 1$.)

1. Let $u$ be a upper semicontinuous function (USC) in $[0,T]\times\mathbb{R}^N$ and $F$ be lower semicontinuous (otherwise consider the lower semicontinuous envelope $F_*$). Then $u$ is called a **viscosity subsolution** of (1.2), if for all $\varphi \in C^{1,2}\left((0,T]\times\mathbb{R}^N\right)$ the following inequality holds at each local maximum point $(t_0,x_0) \in (0,T]\times\mathbb{R}^N$ of $(u-\varphi)$
$$\varphi_t + F\left(t_0, x_0, u, \mathcal{I}u, D_x\varphi, D_{xx}\varphi\right) \le 0$$
and $u(0,x) \le g(x)$ for $x \in \mathbb{R}^N$.
2. Let $u$ be a lower semicontinuous function (LSC) in $[0,T]\times\mathbb{R}^N$ and $F$ be upper semicontinuous (otherwise consider the upper semicontinuous envelope $F^*$). Then $u$ is called a **viscosity supersolution** of (1.2), if for all $\varphi \in C^{1,2}\left((0,T]\times\mathbb{R}^N\right)$ the following inequality holds at each local minimum point $(t_0,x_0) \in (0,T]\times\mathbb{R}^N$ of $(u-\varphi)$
$$\varphi_t + F\left(t_0, x_0, u, \mathcal{I}u, D_x\varphi, D_{xx}\varphi\right) \ge 0.$$
and $u(0,x) \ge g(x)$ for $x \in \mathbb{R}^N$.
3. $u \in C\left([0,T]\times\mathbb{R}^N\right)$ is said to be a **viscosity solution** of (1.2), if $u$ is both a viscosity subsolution and supersolution of (1.2).

The approach of Benth et al. starts from the free boundary and quasi-variational formulations of the problem. From the quasi-variational formulation, we know that in the stopping region $\mathcal{S}$ the inequality
$$\frac{\partial}{\partial t}V(t,S) + \mathcal{BS}(t,S,V_S,V_{SS})V(t,S) \le 0$$
is valid (cf. [24] or [21]). We want to show that there also exists a lower bound. To do this, we fix a point $(t_0,S_0) \in \mathcal{S}$ and assume

- $V \in C^{1,2}\left([0,T]\times(0,\infty)\right)$,
- $g \in C^2\left((0,\infty)\right)$, $g''(\cdot) \ge 0$ (convex), $|\mathcal{I}[g]| \le C_g$.

Since we also know that $V \ge g$, we suggest that $(t_0,S_0)$ is a local maximizer of $(g-V)$ and therefore we have
$$\frac{\partial}{\partial t}V(t_0,S_0) = 0, \qquad \frac{\partial}{\partial S}V(t_0,S_0) = g'(S_0), \qquad \frac{\partial^2}{\partial S^2}V(t_0,S_0) \ge g''(S_0) \ge 0,$$
so that the following inequality holds
$$\begin{aligned}
0 \ge\ & \frac{\partial}{\partial t}V(t_0,S_0) + \mathcal{BS}(t_0,S_0,V_S,V_{SS})V(t_0,S_0)\\
=\ & \frac{1}{2}S_0^2\tilde{\sigma}^2V_{SS}(t_0,S_0) + (r(t_0)-q(t_0,S_0))S_0g'(S_0) - r(t_0)g(S_0) + \mathcal{I}\left[g\right]\\
\ge\ & (r(t_0)-q(t_0,S_0))S_0g'(S_0) - r(t_0)g(S_0) - C_g\\
=\ & -\left[-(r(t_0)-q(t_0,S_0))S_0g'(S_0) + r(t_0)g(S_0)\right]^+ - C_g \stackrel{\text{def}}{=} -c(t_0,S_0).
\end{aligned}$$

**Theorem 1.** *The American option valuation problem is equivalent to the existence and uniqueness of a viscosity solution of the problem*

$$\begin{cases} V_t(t,S) + \mathcal{BS}(t,S,V_S,V_{SS})V(t,S) = -\zeta(t,S,V(t,S)) & (t,S) \in [0,T) \times (0,\infty) \\ V(T,S) = g(S) & S \in (0,\infty), \end{cases} \tag{1.3}$$

*where* $\zeta$ *is given by*

$$\zeta(t,S,V) \stackrel{def}{=} c(t,S)H(g(S) - V(t,S)), \qquad H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0. \end{cases}$$

*Remark* 1.

1. We note that $V \mapsto \zeta(t,S,V)$ is a discontinuous function. Furthermore, it is nonincreasing which is important for existence and uniqueness of viscosity solutions.
2. Since we have assumed in Definition 1 that the function $F$ is lower semicontinuous in the subsolution case, (1.3) has to be understood in the sense of viscosity solutions and not pointwise as it stands in (1.3), particularly we take the lower semicontinuous envelope $\zeta_*$ of $\zeta$. (analogously for the supersolution case) See [5] for more information.

## §2. Existence and uniqueness

To obtain general results, we consider the general equation (1.2). We denote a typical point in $\Gamma$ by $(t,x,z,I,p,X)$. We assume that the integral has the form

$$\mathcal{I}u(t,x) = \int_{\mathbb{R}^N\setminus\{\mathbf{0}\}} M\left(u(t,x+j(t,x,z)), u(t,x)\right)\, \nu_{t,x}(dz),$$

where $\nu_{t,x}$ are bounded Lévy measures which can depend on $(t,x)$ and satisfy

$$\lim_{(s,y)\to(t,x)} \int_{\mathbb{R}^N\setminus\{\mathbf{0}\}} \varphi(z)\, \nu_{s,y}(dz) = \int_{\mathbb{R}^N\setminus\{\mathbf{0}\}} \varphi(z)\, \nu_{t,x}(dz)$$

for every $\varphi \in C_c\left(\mathbb{R}^N\right)$.

### 2.1. Assumptions

**(I1)** There exists a positive constant $L_M < \infty$ such that

$$\begin{aligned} &(i) && |M(u,v) - M(\tilde{u},\tilde{v})| \leq L_M\left(|u-\tilde{u}| + |v-\tilde{v}|\right); \\ &(ii) && M(u,v) \leq M\left(u-h, v-h\right) \text{ for all } h > 0; \\ &(iii) && M(u,v) \leq M\left(\tilde{u}, v\right) \qquad \text{if } u \leq \tilde{u}. \end{aligned}$$

**(I2)** There exists a constant $\eta > 0$ and a function $\Phi \in \mathcal{B}_{t,x}$, where $\mathcal{B}_{t,x}$ is given by

$$\begin{aligned} \mathcal{B}_{t,x} = \Big\{ &\Phi \in C\left(\mathbb{R}^N\right) \cap L^1\left(\mathbb{R}^N; \nu_{t,x}\right) : \Phi \geq 0, \\ &\lim_{(s,y)\to(t,x)} \int_{\mathbb{R}^N\setminus\{\mathbf{0}\}} \Phi(z)\, \nu_{s,y}(dz) = \int_{\mathbb{R}^N\setminus\{\mathbf{0}\}} \Phi(z)\, \nu_{t,x}(dz) \Big\}, \end{aligned}$$

such that

$$M\left(u(s, y + j(s, y, z)), u(s, y)\right) \le \Phi(z)$$

for all $(s, y) \in B_\eta(t, x)$, $z \in \mathbb{R}^N$ and bounded functions $u$.

**(I3)** There exists a positive constant $C_M = C_M(\nu) < \infty$ such that

$$(i) \qquad \int_{\mathbb{R}^N \setminus \{\mathbf{0}\}} \max\left\{1, |j(t, x, z)|^2\right\} \nu_{t,x}(dz) \le C_M;$$

$$(ii) \qquad \int_{\mathbb{R}^N \setminus \{\mathbf{0}\}} |j(t, x, z) - j(s, y, z)|^2 \nu_{t,x}(dz) \le C_M \left(|x - y|^2 + |t - s|^2\right);$$

$$(iii) \qquad \int_{\mathbb{R}^N \setminus \{\mathbf{0}\}} M\left(u(t, x + j(t, x, z)), u(t, x)\right) \left[\nu_{t,x} - \nu_{s,y}\right](dz) \le C_M \left(|x - y| + |t - s|\right).$$

**(A1)** $F$ is **degenerate elliptic**, i.e. $F$ satisfies

$$F(t, x, z, I, p, X + Y) \le F(t, x, z, I, p, X) \qquad \forall Y \ge 0.$$

**(A2)** $F$ is **proper**, i.e. $F$ satisfies

$$F(t, x, z_1, I, p, X) \le F(t, x, z_2, I, p, X) \qquad \text{and} \qquad F(t, x, z, I_1, p, X) \ge F(t, x, z, I_2, p, X)$$

for all $(t, x) \in [0, T] \times \mathbb{R}^N$, $z, z_1, z_2, I, I_1, I_2 \in \mathbb{R}$ with $z_1 \le z_2$ and $I_1 \le I_2$, $p \in \mathbb{R}^N$, $X \in \mathbb{S}^N$.

**(A3)** Suppose that $X, Y \in \mathbb{S}^N$ satisfy

$$-\mu \begin{pmatrix} Id & 0 \\ 0 & Id \end{pmatrix} \le \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \le \lambda_1 \begin{pmatrix} Id & -Id \\ -Id & Id \end{pmatrix} + \lambda_2 \begin{pmatrix} Id & 0 \\ 0 & Id \end{pmatrix}$$

where $\lambda_1 > 0$ and $\mu, \lambda_2 \ge 0$. Then there exists a positive constant $C_{(A3)}$ such that

$$\begin{aligned} F(t, x, z, I_1, p_1, X) - &F(t, x, z, I_2, p_2, X) \\ &\le C_{(A3)} \left[(1 + |x|)\,|p_1 - p_2| + \lambda_1 |p_1 - p_2|^2 + (I_2 - I_1)\right] \end{aligned}$$

for every $(t, x, z, I_i, p_i, X) \in \Gamma$ $(i = 1, 2)$, and

$$\begin{aligned} F(t, x, z, I, p, Y) &- F(t, y, z, I, p, X) \\ &\le C_{(A3)} \lambda_1 |x - y|^2 + \lambda_2 C_{(A3)} \left(1 + |x|^2 + |y|^2\right) + C_{(A3)} (1 + |p|)\,|x - y| \end{aligned}$$

for every $(t, x, z, I, p, Y), (t, y, z, I, p, X) \in \Gamma$.

**(A4)** There exists a positive constant $C = C(z, p, X) < \infty$ such that

$$|F(t, x, z, I, p, X)| \le C\left(1 + |z| + |p| + \|X\|\right)$$

for all $(t, x, z, I, p, X) \in \Gamma$.

## 2.2. Existence

The proof of existence goes back to Perron (cf. [23]), who used this method to prove existence for the Laplace problem.

**Theorem 2.** *Let $F$ satisfy (A1), (A2), (I1) and let $F$ be USC in the supersolution case as well as LSC in the subsolution case. Moreover, we assume that $g \in C\left(\mathbb{R}^N\right)$ and*

1. $\underline{u}, \overline{u} \in C\left([0,T] \times \mathbb{R}^N\right)$ *are respectively viscosity sub- and supersolutions of (1.2) such that* $\underline{u} \leq \overline{u}$ *on* $[0,T] \times \mathbb{R}^N$*;*
2. *for every* $(t,x) \in [0,T] \times \mathbb{R}^N$ *there exists a positive constant* $\eta$ *and a function* $\Phi \in \mathcal{B}_{t,x}$ *such that*

$$-\Phi(z) \leq M\left(\underline{u}(s, y + j(s,y,z)), w\right) \leq M\left(\overline{u}(s, y + j(s,y,z)), w\right) \leq \Phi(z)$$

*for all* $(s,y) \in B_\eta(t,x)$ $(\eta > 0)$*,* $z \in \mathbb{R}^N$ *and* $w \in \left[\underline{u}(s,y), \overline{u}(s,y)\right]$.

*Furthermore, let* $\mathfrak{S}$ *be the set of all subsolutions* $v$ *of (1.2), such that* $\underline{u} \leq v \leq \overline{u}$ *on* $[0,T] \times \mathbb{R}^N$*. If we set*

$$u(t,x) \overset{def}{=} \sup\left\{v(t,x) : v \in \mathfrak{S}\right\},$$

*then the upper semicontinuous envelope* $u^*$ *is a subsolution of (1.2) and the lower semicontinuous envelope* $u_*$ *is a supersolution of (1.2).*

*Proof.* The result follows from [1, Proof of Proposition 1] and [2, Proof of Theorem 3.3].

□

## 2.3. Uniqueness

**Theorem 3.** *Let* $\underline{u} \in USC\left([0,T] \times \mathbb{R}^N\right)$*,* $\overline{u} \in LSC\left([0,T] \times \mathbb{R}^N\right)$ *be respectively a bounded viscosity sub- and a supersolution of (1.2). Furthermore, let $F$ be continuous and satisfy the assumptions (A1), (A2), (A3), (I1)-(I3). Moreover, let* $\underline{u}(0,x)$ *and* $\overline{u}(0,x)$ *be uniformly continuous with modulus of continuity* $m(\cdot)$*. Then*

$$\sup_{[0,T]\times\mathbb{R}^N} \left(\underline{u} - \overline{u}\right) \leq e^{(c_0+1)T} \sup_{x\in\mathbb{R}^N} \left(\underline{u}(0,x) - \overline{u}(0,x)\right)^+ \tag{2.1}$$

*where* $c_0$ *is a positive constant.*

*Proof.* The proof is based on [11] and [25, Proof of Theorem 1.2.1]. The first parts can be proven analogously. In the 9. step in [25, Proof of Theorem 1.2.1], by using assumption (A3), one gets the additional term $\mathcal{I}\overline{u}\,(t_0, y_0) - \mathcal{I}\underline{u}\,(t_0, x_0)$ (we have changed the notation from $\left(\bar{t}, \overline{x}\right)$ to $(t_0, x_0)$). For this remaining integral term, we take the limit as $\epsilon \searrow 0$ (by using (I2)) and obtain

$$
\begin{aligned}
& \mathcal{I}\overline{u}\,(t_0,x_0) - \mathcal{I}\underline{u}\,(t_0,x_0) \\
= & \int \Big[M\left(\overline{u}\,(t_0,x_0+j\,(t_0,x_0,z)),\overline{u}\,(t_0,x_0)\right) \\
& \qquad -M\left(\underline{u}\,(x_0+j\,(t_0,x_0,z)),\underline{u}\,(t_0,x_0)\right)\Big]\,\nu_{t_0,x_0}\,(dz) \\
\overset{(I1)}{\geq} & \int \Big[M\left(\overline{u}\,(t_0,x_0+j\,(t_0,x_0,z)),\overline{u}\,(t_0,x_0)\right) \\
& \qquad -M\left(\underline{u}\,(x_0+j\,(t_0,x_0,z))-\underline{u}\,(t_0,x_0)+\overline{u}\,(t_0,x_0),\overline{u}\,(t_0,x_0)\right)\Big]\,\nu_{t_0,x_0}\,(dz)
\end{aligned}
$$

where we have used that $\underline{u}\,(t_0,x_0) \geq \overline{u}\,(t_0,x_0)$. Next, we use that

$$
\underline{u}\,(t_0,x_0) - \overline{u}\,(t_0,x_0) - \delta\,|x_0|^2 \geq \underline{u}(t,x) - \overline{u}(t,x) - \delta\,|x|^2
$$

for all $(t,x) \in [0,T] \times \mathbb{R}^N$. Choosing $(t,x) = (t_0, x_0 + j\,(t_0,x_0,z))$, we obtain

$$
\begin{aligned}
& \underline{u}\,(t_0,x_0+j\,(t_0,x_0,z)) - \underline{u}\,(t_0,x_0) + \overline{u}\,(t_0,x_0) \\
\leq & \,\overline{u}\,(t_0,x_0+j\,(t_0,x_0,z)) + 2\delta\,|x_0 j\,(t_0,x_0,z)| + \delta\,|j\,(t_0,x_0,z)|^2\,.
\end{aligned}
\tag{2.2}
$$

Using condition (I3), we further have

$$
\begin{aligned}
& \int \Big[M\left(\overline{u}\,(t_0,x_0+j\,(t_0,x_0,z)),\overline{u}\,(t_0,x_0)\right) \\
& \qquad - M\left(\underline{u}\,(t_0,x_0+j\,(t_0,x_0,z))-\underline{u}\,(t_0,x_0)+\overline{u}\,(t_0,x_0),\overline{u}\,(t_0,x_0)\right)\Big]\nu_{t_0,x_0}\,(dz) \\
\overset{(2.2)}{\geq} & \int \Big[M\left(\overline{u}\,(t_0,x_0+j\,(t_0,x_0,z)),\overline{u}\,(t_0,x_0)\right) \\
& - M\left(\overline{u}\,(t_0,x_0+j\,(t_0,x_0,z))+2\delta|x_0 j(t_0,x_0,z)|+\delta|j(t_0,x_0,z)|^2,\overline{u}\,(t_0,x_0)\right)\Big]\nu_{t_0,x_0}\,(dz)\,.
\end{aligned}
$$

Next, we estimate the integral

$$
\begin{aligned}
& \int \Big[M\left(\overline{u}\,(t_0,x_0+j\,(t_0,x_0,z)),\overline{u}\,(t_0,x_0)\right) \\
& - M\left(\overline{u}\,(t_0,x_0+j\,(t_0,x_0,z))+2\delta|x_0 j(t_0,x_0,z)|+\delta|j(t_0,x_0,z)|^2,\overline{u}\,(t_0,x_0)\right)\Big]\nu_{t_0,x_0}\,(dz) \\
\overset{(I1)}{\geq} & -\delta \int 2\,|x_0 j\,(t_0,x_0,z)| + |j\,(t_0,x_0,z)|^2\;\nu(dz) \\
\geq & -\delta \int \left[2\,|x_0|\left(\frac{1}{2}+\frac{|j\,(t_0,x_0,z)|^2}{2}\right)+|j\,(t_0,x_0,z)|^2\right]\nu(dz) \\
\overset{(I3)}{\geq} & -2C\delta\,|x_0| \xrightarrow{\delta\searrow 0} 0,
\end{aligned}
$$

where $C = C(C_M) < \infty$ is a suitable positive constant. Using the arguments of [25, Proof of Theorem 1.2.1], one can show that the desired result follows.

□

## §3. American option problem

Now, we want to put this formulation into the form (1.2) and obtain that the function $F$ is given by

$$
\begin{aligned}
F_{AO}(t,S,z,I,p,X) = &\ F_{AO}^{cont}(t,S,z,I,p,X) + \zeta(t,S,z)\\
= &\ \frac{1}{2}\tilde{\sigma}^2(t,S,p,X)S^2X + (r(t) - q(t,S))Sp - r(t)z + I\\
&+ c(t,S)H(g(S) - z).
\end{aligned}
$$

Performing the backward time ($\tau = T - t$) and Euler-transformation ($S = e^x$) yields

$$
\begin{aligned}
\tilde{F}_{AO}(\tau,x,z,I,p,X) = &\ \tilde{F}_{AO}^{cont}(\tau,x,z,I,p,X) - \hat{\zeta}(\tau,x,z)\\
= &\ -\frac{1}{2}\hat{\tilde{\sigma}}^2(\tau,x,p,X)(X-p) - (\hat{r}(\tau) - \hat{q}(\tau,x))\,p + \hat{r}(\tau)z - I\\
&- \hat{c}(\tau,x)H(\hat{g}(x) - z),
\end{aligned}
$$

where $\hat{(\cdot)}$ denotes the transformed version of $(\cdot)$. Next, we want to prove existence and uniqueness.

**Theorem 4.** *Let $\tilde{F}_{AO}^{cont}$ be continuous and satisfy (A1)-(A4), (I1)-(I3). Furthermore, let $g$ be convex, $\hat{g} \in W^{2,\infty}(\mathbb{R})$ and $\hat{r}, \hat{q} \in C_b$. Then there exists a unique viscosity solution of the American option valuation problem (1.3).*

*Proof.* In order to prove this result, we have to check that we can use Perron's method, the comparison principle and that there exists a sub- and supersolution of the problem.

1. <u>existence:</u> Theorem 2 does not require the continuity of $F$, but parabolicity and semicontinuity (USC for the supersolution and LSC for the subsolution formulation). Since $\tilde{F}_{AO}$ differs from the usual models only by the nonlinearity $\hat{\zeta}$, which does not depend on the second derivative, the parabolicity is clear. Since $\hat{\zeta}$ is upper semicontinuous in the supersolution and lower semicontinuous in the subsolution formulation, the requirements of Perron's method are satisfied.
2. <u>uniqueness:</u> Because of the nonlinearity, we have to prove the comparison principle again. At first, we prove that $\tilde{F}_{AO}$ is proper which is essentially for the comparison principle. Therefore, we focus our attention on the term

$$-\hat{c}(\tau,x)H(\hat{g}(x) - z).$$

Because $\hat{c}(\tau,x) \geq 0$ and the Heaviside function $H$ is monotonously nondecreasing, we observe that this term is also nondecreasing in $z$. Hence, $\tilde{F}_{AO}$ satisfies assumption (A2). If we consider the proof of Theorem 3 again, we observe that we have to deal with the additional term

$$
\begin{aligned}
\tilde{\mathfrak{A}} = &\ \left(\hat{\zeta}^*\left(\tau_0, y_0, \underline{u}(\tau_0,x_0)\right) - \hat{\zeta}_*\left(\tau_0, x_0, \underline{u}(\tau_0,x_0)\right)\right)\\
= &\ \left(\hat{c}(\tau_0,y_0)\,H^*\left(\hat{g}(y_0) - \underline{u}(\tau_0,x_0)\right) - \hat{c}(\tau_0,x_0)\,H_*\left(\hat{g}(x_0) - \underline{u}(\tau_0,x_0)\right)\right),
\end{aligned}
$$

where $\hat{c}(\tau, x) = c(T - \tau, e^x)$. Furthermore, we have

$$\begin{aligned} & \hat{c}(\tau_0, y_0) H^* \left(\hat{g}(y_0) - \underline{u}(\tau_0, x_0)\right) - \hat{c}(\tau_0, x_0) H_* \left(\hat{g}(x_0) - \underline{u}(\tau_0, x_0)\right) \\ = & \begin{cases} \hat{c}(\tau_0, y_0) - \hat{c}(\tau_0, x_0) & \text{if } \hat{g}(y_0) \geq \underline{u}(\tau_0, x_0) \text{ and } \hat{g}(x_0) > \underline{u}(\tau_0, x_0) \\ \hat{c}(\tau_0, y_0) & \text{if } \hat{g}(y_0) \geq \underline{u}(\tau_0, x_0) \text{ and } \hat{g}(x_0) \leq \underline{u}(\tau_0, x_0) \\ -\hat{c}(\tau_0, x_0) & \text{if } \hat{g}(y_0) < \underline{u}(\tau_0, x_0) \text{ and } \hat{g}(x_0) > \underline{u}(\tau_0, x_0) \\ 0 & \text{if } \hat{g}(y_0) < \underline{u}(\tau_0, x_0) \text{ and } \hat{g}(x_0) \leq \underline{u}(\tau_0, x_0). \end{cases} \end{aligned}$$

The third case is irrelevant, because in that case we would have $\hat{g}(y_0) < \underline{u}(\tau_0, x_0) < \hat{g}(x_0)$. From the continuity of $\hat{g}$ follows that $\hat{g}(y_0) = \underline{u}(\tau_0, x_0) = \hat{g}(x_0)$ for $\epsilon, \delta \searrow 0$ and we would be back in the second case. Hence

$$\begin{aligned} & \min\{0, \hat{c}(\tau_0, y_0) - \hat{c}(\tau_0, x_0)\} \\ \leq \quad & \hat{c}(\tau_0, y_0) H^* \left(\hat{g}(y_0) - \underline{u}(\tau_0, x_0)\right) - \hat{c}(\tau_0, x_0) H_* \left(\hat{g}(x_0) - \underline{u}(\tau_0, x_0)\right) \leq \hat{c}(\tau_0, y_0). \end{aligned}$$

Because $\hat{c}$ is continuous and positive, it follows that $\hat{c}(\tau_0, x_0) \xrightarrow{\delta, \epsilon \searrow 0} \hat{c}(\tau_0, y_0)$ and thus $\mathfrak{A} \geq 0$. Therefore, the proof of the comparison principle also works if we add the discontinuity $\hat{\zeta}$.

3. <u>construction of a sub- and supersolution:</u> To construct a sub- and supersolution, we use that $\hat{g} \in W^{2,\infty}(\mathbb{R})$ and (A4) holds true. If we consider the transformed problem $(u(\tau, x) = V(T - \tau, e^x) = V(t, S))$ where $\hat{\zeta}$ is given by

$$\begin{aligned} \hat{\zeta}(\tau, x, u(\tau, x)) = \quad & \hat{c}(\tau, x) H\left(\hat{g}(x) - u(\tau, x)\right) \\ = \quad & \left[\left(\hat{r}(\tau) - \hat{q}(\tau, x)\right) \hat{g}_x(x) + \hat{r}(\tau)\hat{g}(x)\right]^+ H\left(\hat{g}(x) - u(\tau, x)\right), \end{aligned}$$

we observe that $\hat{\zeta}$ is bounded since $\hat{r}$, $\hat{q}$ are bounded and $\hat{g} \in W^{2,\infty}(\mathbb{R})$. Therefore, a sub- and supersolution can be constructed by

$$\underline{u}(\tau, x) = \hat{g}(x) - C\tau \qquad \text{and} \qquad \overline{u}(\tau, x) = \hat{g}(x) + C\tau,$$

where

$$C = \sup_{[0,T]\times\mathbb{R}} \left(\left|\tilde{F}_{AO}^{cont}(\tau, x, \hat{g}(x), \hat{g}_x(x), \hat{g}_{xx}(x))\right| + \left|\hat{\zeta}(\tau, x, \hat{g}(x))\right|\right) \overset{\hat{g} \in W^{2,\infty}}{<} \infty.$$

□

*Remark* 2.

1. We note that a viscosity solution $V_{Eur}$ of the European option problem satisfies

$$\begin{cases} \dfrac{\partial}{\partial t} V_{Eur}(t, S) + \mathcal{BS}(t, S, V_S, V_{SS}) V_{Eur}(t, S) = 0 & (t, S) \in [0, T) \times (0, \infty) \\ V_{Eur}(T, S) = g(S) & S \in (0, \infty), \end{cases}$$

in the viscosity sense. Furthermore, we know that $\zeta \geq 0$. Hence, we can conclude that the viscosity solution of the European option pricing problem is also a subsolution of the American option pricing problem, in particular, we have

$$\begin{cases} \dfrac{\partial}{\partial t} V_{Eur}(t,S) + \mathcal{BS}(t,S,V_S,V_{SS})V_{Eur}(t,S) \geq -\zeta(t,S,V_{Eur}(t,S)) \\ \qquad\qquad\qquad\qquad\qquad\qquad V_{Eur}(T,S) \leq g(S), \end{cases}$$

for $(t,S) \in [0,T) \times (0,\infty)$ in the viscosity sense. This result is not very surprising, because an American option includes a European option and therefore the price must be higher.

2. Under the assumptions of Theorem 4, one can prove existence and uniqueness of a viscosity solution of the American option pricing problem for the models mentioned in the introduction.

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