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Oseen Problem in $\mathbb{R}^3$: An Approach in Weighted Sobolev Spaces

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Abstract. In this paper, we study the Oseen problem in $\mathbb{R}^3$. We are interested in the existence and uniqueness of generalized solutions and we give some regularity results. Our approach rests on the use of weighted Sobolev spaces.

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§1. Introduction

We consider the Oseen problem in $\mathbb{R}^3$: For a given vector field $\boldsymbol{f}$ and a scalar function h, we look for a velocity field $\boldsymbol{u}$ and a pressure π which fulfil:

$$-\Delta \boldsymbol{u} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{u}) + \nabla \pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = h \quad \text{in} \quad \mathbb{R}^3, \tag{1}$$

where, $\boldsymbol{v}$ is a given velocity field belonging to $\boldsymbol{L}^3(\mathbb{R}^3)$ with divergence free. The existence and uniqueness of problem (1) are well known in the classical Sobolev spaces $\boldsymbol{W}^{m,p}(\Omega)$ when the domain Ω is bounded *i.e* with a boundary condition. It is well known that it is not possible to extend this result to the case of unbounded domains in which we are interested, here the spaces $\boldsymbol{W}^{m,p}(\Omega)$ is not adequate. Therefore, a specific functional framework is necessary which also has to take into account the behaviour of the functions at infinity.

§2. Basic concepts on weighted Sobolev spaces

Let $x = (x_1, x_2, x_3)$ be a typical point in $\mathbb{R}^3$ and let $r = |x| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$ denote its distance to the origin. In order to control the behaviour at infinity of our functions and distributions we use for basic weight the quantity $\rho(x) = (1 + r^2)^{1/2}$ which is equivalent to r at infinity, and to one on any bounded subset of $\mathbb{R}^3$. We define $\mathcal{D}(\mathbb{R}^3)$ to be the linear space of infinite differentiable functions with compact support on $\mathbb{R}^3$. Now, let $\mathcal{D}'(\mathbb{R}^3)$ denote the dual space of $\mathcal{D}(\mathbb{R}^3)$, often called the space of distributions on $\mathbb{R}^3$. We denote by $< .,. >$ the duality pairing between $\mathcal{D}'(\mathbb{R}^3)$ and $\mathcal{D}(\mathbb{R}^3)$. For each $p \in \mathbb{R}$ and $1 < p < \infty$, the conjugate exponent p' is given by the relation $\frac{1}{p} + \frac{1}{p'} = 1$. Then, for any non-negative integers m and real numbers $p > 1$ and α, setting

$$k = k(m, p, \alpha) = \begin{cases} -1, & \text{if} \quad \frac{3}{p} + \alpha \notin \{1, ..., m\}, \\ m - \frac{3}{p} - \alpha, & \text{if} \quad \frac{3}{p} + \alpha \in \{1, ..., m\}, \end{cases}$$

we define the following space:

$$\begin{aligned} W^{m,p}_{\alpha}(\mathbb{R}^3) = \{ & u \in \mathcal{D}'(\mathbb{R}^3); \\ & \forall\, \lambda \in \mathbb{N}^3 : \ 0 \le |\lambda| \le k,\ \rho^{\alpha-m+|\lambda|}(\ln(1+\rho))^{-1} D^{\lambda} u \in L^p(\mathbb{R}^3); \\ & \forall\, \lambda \in \mathbb{N}^3 : \ k+1 \le |\lambda| \le m, \rho^{\alpha-m+|\lambda|} D^{\lambda} u \in L^p(\mathbb{R}^3)\}. \end{aligned}$$

It is a reflexive Banach space equipped with its natural norm:

$$\|u\|_{W^{m,p}_{\alpha}(\mathbb{R}^3)} = \Bigg(\sum_{0 \le |\lambda| \le k} \|\rho^{\alpha-m+|\lambda|}(\ln(1+\rho))^{-1} D^{\lambda} u\|^p_{L^p(\mathbb{R}^3)} + \sum_{k+1 \le |\lambda| \le m} \|\rho^{\alpha-m+|\lambda|} D^{\lambda} u\|^p_{L^p(\mathbb{R}^3)} \Bigg)^{1/p}.$$

For $m = 0$, we set

$$W^{0,p}_{\alpha}(\mathbb{R}^3) = \{u \in \mathcal{D}'(\mathbb{R}^3);\ \ \rho^{\alpha}\, u \in L^p(\mathbb{R}^3)\}.$$

We note that the logarithmic weight only appears if $p = 3$ or $p = \frac{3}{2}$ and all the local properties of $W^{m,p}_{\alpha}(\mathbb{R}^3)$ coincide with those of the classical Sobolev space $W^{m,p}(\mathbb{R}^3)$. We set $W^{m,p}_{\alpha}(\mathbb{R}^3)$ as the adherence of $\mathcal{D}(\mathbb{R}^3)$ for the norm $\|\ .\ \|_{W^{m,p}_{\alpha}(\mathbb{R}^3)}$. Then, the dual space of $W^{m,p}_{\alpha}(\mathbb{R}^3)$, denoting by $W^{-m,p'}_{-\alpha}(\mathbb{R}^3)$, is a space of distributions. On the other hand, these spaces obey the following imbedding

$$W^{m,p}_{\alpha}(\mathbb{R}^3) \hookrightarrow W^{m-1,p}_{\alpha-1}(\mathbb{R}^3)$$

if and only if $m > 0$ and $3/p + \alpha \neq 1$ or $m \le 0$ and $3/p + \alpha \neq 3$. In addition, we have for $\alpha = 0$ or $\alpha = 1$

$$W^{1,p}_{\alpha}(\mathbb{R}^3) \hookrightarrow W^{0,p*}_{\alpha}(\mathbb{R}^3) \quad \text{where} \quad p* = \frac{3p}{3-p} \quad \text{and} \quad 1 < p < 3. \tag{2}$$

Consequently, by duality, we have

$$W^{0,q}_{-\alpha}(\mathbb{R}^3) \hookrightarrow W^{-1,p'}_{-\alpha}(\mathbb{R}^3) \quad \text{where} \quad q = \frac{3p'}{3+p'} \quad \text{and} \quad p' > 3/2.$$

Moreover, the Hardy inequality holds,

$$\forall u \in W^{1,p}_{\alpha}(\mathbb{R}^3), \quad \begin{cases} \|u\|_{W^{1,p}_{\alpha}(\mathbb{R}^3)} \le C \|\nabla\, u\|_{\mathbf{W}^{0,p}_{\alpha}(\mathbb{R}^3)}, & \text{if} \quad 3/p + \alpha > 1, \\ \|u\|_{W^{1,p}_{\alpha}(\mathbb{R}^3)/\mathcal{P}_0} \le C \|\nabla\, u\|_{\mathbf{W}^{0,p}_{\alpha}(\mathbb{R}^3)}, & \text{otherwise}, \end{cases}$$

where $\mathcal{P}_0$ stands for the space of constant functions in $W^{1,p}_{\alpha}(\mathbb{R}^3)$ when $3/p + \alpha \le 1$ and C satisfies $C = C(p, \alpha) > 0$.

§3. Generalized solutions in $\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)$.

We are interested in the existence and the uniqueness of generalized solutions $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$, with $1 < p < \infty$, to the Problem (1). We will consider the following data:

$$\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^3), \quad \boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3) \quad \text{and} \quad h \in L^p(\mathbb{R}^3).$$

On the one hand if $\boldsymbol{u} \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3)$, then we have $\boldsymbol{u} \in \boldsymbol{L}^{3/2}_{loc}(\mathbb{R}^3)$ and thus $\boldsymbol{v} \otimes \boldsymbol{u}$ belongs to $\boldsymbol{L}^1_{loc}(\mathbb{R}^3)$. It means that $\operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{u})$ is well defined as a distribution in $\mathbb{R}^3$. On the other hand, if $p \geq 3/2$, we deduce that the term $\boldsymbol{v} \cdot \nabla \boldsymbol{u}$ is well defined and we can write $\operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{u}) = \boldsymbol{v} \cdot \nabla \boldsymbol{u}$. Moreover, if $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ with $p < 3$ is a solution to (1), we have for any $\boldsymbol{\varphi} \in \boldsymbol{\mathcal{D}}(\mathbb{R}^3)$:

$$\int_{\mathbb{R}^3} ((\nabla \boldsymbol{u} + \boldsymbol{v} \otimes \boldsymbol{u}) : \nabla \boldsymbol{\varphi} - \pi \operatorname{div} \boldsymbol{\varphi}) = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_{W_0^{-1,p}(\mathbb{R}^3) \times W_0^{1,p'}(\mathbb{R}^3)} . \tag{3}$$

Observe that in this case, $\boldsymbol{u} \in \boldsymbol{L}^{p*}(\mathbb{R}^3)$ with $\frac{1}{p*} = \frac{1}{p} - \frac{1}{3}$, so $\boldsymbol{v} \otimes \boldsymbol{u} \in \boldsymbol{L}^p(\mathbb{R}^3)$. Because $\boldsymbol{\mathcal{D}}(\mathbb{R}^3)$ is dense in $\boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)$, this last relation holds for any $\boldsymbol{\varphi} \in \boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)$. As this last space contains the constant vectors when $p' \geq 3$, the force $\boldsymbol{f}$ must satisfies the following compatibility condition:

$$\langle f_i, 1 \rangle_{W_0^{-1,p}(\mathbb{R}^3) \times W_0^{1,p'}(\mathbb{R}^3)} = 0 \quad \text{for any } i = 1, 2, 3 \quad \text{if} \quad p \leq 3/2. \tag{4}$$

If $p \geq 3$, (1) is equivalent to the following variational problem:

$$\int_{\mathbb{R}^3} (\nabla \boldsymbol{u} : \nabla \boldsymbol{\varphi} - \pi \operatorname{div} \boldsymbol{\varphi} + \boldsymbol{v} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{\varphi}) = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_{W_0^{-1,p}(\mathbb{R}^3) \times W_0^{1,p'}(\mathbb{R}^3)} . \tag{5}$$

Remark 1. To simplify the study of problem (1), we can suppose at first that $h = 0$. Indeed, if h in $L^p(\mathbb{R}^3)$, there exists $\chi \in W_0^{2,p}(\mathbb{R}^3)$ such that $\Delta \chi = h$ (see [1]) and satisfying

$$\|\nabla \chi\|_{W_0^{1,p}(\mathbb{R}^3)} \leqslant C \|h\|_{L^p(\mathbb{R}^3)}.$$

Set $\boldsymbol{w}_h = \nabla \chi \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3)$ and $\boldsymbol{z} = \boldsymbol{u} - \boldsymbol{w}_h$. Then problem (1) becomes:

$$-\Delta \boldsymbol{z} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{z}) + \nabla \pi = \boldsymbol{f} + \Delta \boldsymbol{w}_h - \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{w}_h) \quad \text{and} \quad \operatorname{div} \boldsymbol{z} = 0 \quad \text{in} \quad \mathbb{R}^3.$$

If $1 < p < 3$, we have $\boldsymbol{w}_h \in \boldsymbol{L}^{p*}(\mathbb{R}^3)$ and $\boldsymbol{v} \otimes \boldsymbol{w}_h$ belongs to $\boldsymbol{L}^p(\mathbb{R}^3)$. Consequently $\operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{w}_h)$ belongs to $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)$. However when $p \geq 3$, $\operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{w}_h) = \boldsymbol{v} \cdot \nabla \boldsymbol{w}_h$ belongs to $\boldsymbol{L}^r(\mathbb{R}^3)$, with $\frac{1}{r} = \frac{1}{3} + \frac{1}{p}$ and $\boldsymbol{L}^r(\mathbb{R}^3) \hookrightarrow \boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)$. This means that $\mathbf{F} := \boldsymbol{f} + \Delta \boldsymbol{w}_h - \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{w}_h)$ belongs to $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)$. In addition, we have for any $i = 1, 2, 3$ and $p \leqslant \frac{3}{2}$ the equivalence

$$\langle f_i, 1 \rangle_{W_0^{-1,p}(\mathbb{R}^3) \times W_0^{1,p'}(\mathbb{R}^3)} = 0 \Longleftrightarrow \langle F_i, 1 \rangle_{W_0^{-1,p}(\mathbb{R}^3) \times W_0^{1,p'}(\mathbb{R}^3)} = 0. \tag{6}$$

This means that to solve (1), it is sufficient to solve the following problem:

$$-\Delta \boldsymbol{u} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{u}) + \nabla \pi = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 \quad \text{in} \quad \mathbb{R}^3. \tag{7}$$

In the following theorem, we establishes the existence of generalized solutions to Problem (1) in the case $1 < p \leq 2$. The uniqueness of the solutions will be studied later.

Theorem 1. *Let* $1 < p \leqslant 2$. *Assume that* $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)$ *satisfies the compatibility condition* (4) *and let* $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$. *Then the Oseen problem* (7) *has a solution* $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ *such that*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + \|\pi\|_{L^p(\mathbb{R}^3)} \leqslant C(1 + \|\boldsymbol{v}\|_{\boldsymbol{L}^3(\mathbb{R}^3)})\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}. \tag{8}$$

Proof. First, the case $p = 2$ is an immediate consequence of the following property

$$\forall \boldsymbol{w} \in \boldsymbol{W}_0^{1,2}(\mathbb{R}^3), \qquad \int_{\mathbb{R}^3} (\boldsymbol{v} \cdot \nabla)\, \boldsymbol{w} \, \cdot \boldsymbol{w} = 0$$

and Lax-Milgram aligna. So we can suppose that $1 < p < 2$.

The main idea of the proof is to observe that $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$ can be approximated by a smooth function $\boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}_\sigma(\mathbb{R}^3)$. Given ε, there is $\boldsymbol{\psi}_\epsilon \in \boldsymbol{\mathcal{D}}_\sigma(\mathbb{R}^3)$ such that

$$\|\boldsymbol{v} - \boldsymbol{\psi}_\varepsilon\|_{\boldsymbol{L}^3(\mathbb{R}^3)} < \varepsilon, \tag{9}$$

where $\varepsilon > 0$ is a constant which will be fixed as below. By (4) and [3], we have $\boldsymbol{f} = \operatorname{div} \mathbf{F}$ with $\mathbf{F} \in \boldsymbol{L}^p(\mathbb{R}^3)$. Let $\rho \in \mathcal{D}(\mathbb{R}^3)$, be a smooth C^∞ function with compact support in $B(0, 1)$, such that $\rho \geq 0$, $\int_{\mathbb{R}^3} \rho(x)\, dx = 1$. For $t \in (0, 1)$, let ρ_t denote the function $x \longmapsto (\frac{1}{t^3})\rho(\frac{x}{t})$. Let $\varphi \in \mathcal{D}(\mathbb{R}^3)$ such that $0 \leqslant \varphi(x) \leqslant 1$ for any $x \in \mathbb{R}^3$, and

$$\varphi(x) = \begin{cases} 1 & \text{if} \quad 0 \leqslant | x | \leqslant 1, \\ 0 & \text{if} \quad | x | \geqslant 2. \end{cases}$$

We begin by applying the cut off functions φ_k, defined on $\mathbb{R}^3$ for any $k \in \mathbb{N}^*$, as $\varphi_k(x) = \varphi(\frac{x}{k})$. Set $\mathbf{F}_k = \varphi_k \mathbf{F}$. Thus we obtain

$$\mathbf{G}_{t,k} = \rho_t * \mathbf{F}_k \in \mathcal{D}(\mathbb{R}^3) \quad \text{and} \quad \lim_{t \to 0} \lim_{k \to \infty} \mathbf{G}_{t,k} = \mathbf{F} \quad \text{in} \quad \boldsymbol{L}^p(\mathbb{R}^3). \tag{10}$$

Now, observe that using Young inequality, we have

$$\|\rho_t * \mathbf{F}_k\|_{\boldsymbol{L}^2(\mathbb{R}^3)} \leqslant \|\rho_t\|_{\boldsymbol{L}^q(\mathbb{R}^3)} \|\mathbf{F}_k\|_{\boldsymbol{L}^p(\mathbb{R}^3)}, \tag{11}$$

with $q = \dfrac{2p}{3p - 2}$. Observe that $q > 1$ is equivalent to $p < 2$. After an easy calculation, we obtain that

$$\|\rho_t * \mathbf{F}_k\|_{\boldsymbol{L}^2(\mathbb{R}^3)} \leqslant \frac{4}{3}\pi t^{\frac{-3}{q'}} \|\mathbf{F}_k\|_{\boldsymbol{L}^p(\mathbb{R}^3)}. \tag{12}$$

We choose $t = k^{-\alpha}$ with $\alpha > 0$ which will be precise later. Set now $\boldsymbol{f}_k = \operatorname{div} \mathbf{G}_{t,k}$ for any $k \in \mathbb{N}^*$. Then we have

$$\boldsymbol{f}_k \to \boldsymbol{f} \quad \text{in} \quad \boldsymbol{W}_0^{-1,p}(\mathbb{R}^3).$$

It is clear that $\boldsymbol{f}_k$ satisfies the condition (4).

Step 1. We suppose that $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}_\sigma(\mathbb{R}^3)$. Thanks to Lemma 4.1 see [2], there exists a unique solution

$$\boldsymbol{u}_k \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,2}(\mathbb{R}^3), \quad \pi_k \in L^2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$$

satisfying

$$-\Delta\, \boldsymbol{u}_k + \operatorname{div}(\boldsymbol{v}\otimes \boldsymbol{u}_k) + \nabla\, \pi_k = \boldsymbol{f}_k, \quad \operatorname{div} \boldsymbol{u}_k = 0 \quad \text{in} \quad \mathbb{R}^3. \tag{13}$$

Set $B_\varepsilon = \operatorname{supp} \boldsymbol{\psi}_\varepsilon$, then from the Stokes theory (see [1] Theorem 3.3), we obtain

$$\|\boldsymbol{u}_k\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + \|\pi_k\|_{L^p(\mathbb{R}^3)} \leqslant C_1 \left(\|\boldsymbol{f}_k\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + \|\boldsymbol{v}\otimes \boldsymbol{u}_k\|_{\boldsymbol{L}^p(\mathbb{R}^3)} \right), \tag{14}$$

where C_1 doesn't depend on k,$\boldsymbol{f}_k$ and $\boldsymbol{v}$. Using Hölder inequality, we have

$$\begin{aligned} \|\boldsymbol{v}\otimes \boldsymbol{u}_k\|_{\boldsymbol{L}^p(\mathbb{R}^3)} &\leqslant \|(\boldsymbol{v} - \boldsymbol{\psi}_\epsilon)\otimes \boldsymbol{u}_k\|_{\boldsymbol{L}^p(\mathbb{R}^3)} + \|\boldsymbol{\psi}_\varepsilon \otimes \boldsymbol{u}_k\|_{\boldsymbol{L}^p(\mathbb{R}^3)} \\ &\leqslant \|\boldsymbol{v} - \boldsymbol{\psi}_\varepsilon\|_{\boldsymbol{L}^3(\mathbb{R}^3)} \|\boldsymbol{u}_k\|_{\boldsymbol{L}^{p^*}(\mathbb{R}^3)} + \|\boldsymbol{\psi}_\varepsilon\|_{\boldsymbol{L}^3(B_\varepsilon)} \|\boldsymbol{u}_k\|_{\boldsymbol{L}^{p^*}(B_\varepsilon)}. \end{aligned} \tag{15}$$

Using the Sobolev inequality, we obtain

$$\|\boldsymbol{u}_k\|_{\boldsymbol{L}^{p^*}(\mathbb{R}^3)} \leqslant C_2 \|\boldsymbol{u}_k\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)}. \tag{16}$$

By the assumption (9), and from (14), (15) and (16) it follows that

$$(1 - C_1 C_2 \varepsilon)\|\boldsymbol{u}_k\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + \|\pi_k\|_{L^p(\mathbb{R}^3)} \leqslant C_1 (\|\boldsymbol{f}_k\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + \|\boldsymbol{\psi}_\varepsilon\|_{\boldsymbol{L}^3(B_\varepsilon)} \|\boldsymbol{u}_k\|_{\boldsymbol{L}^{p^*}(B_\varepsilon)}). \tag{17}$$

Taking $0 < \varepsilon < 1/2C_1C_2$, we obtain

$$\|\boldsymbol{u}_k\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + \|\pi_k\|_{L^p(\mathbb{R}^3)} \leqslant 2C_1 (\|\boldsymbol{f}_k\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + \|\boldsymbol{\psi}_\varepsilon\|_{\boldsymbol{L}^3(B_\varepsilon)} \|\boldsymbol{u}_k\|_{\boldsymbol{L}^{p^*}(B_\varepsilon)}). \tag{18}$$

From (18), we prove that there exists $C > 0$ not depending of k and $\boldsymbol{v}$ such that for any $k \in \mathbb{N}^*$ we have

$$\|\boldsymbol{u}_k\|_{\boldsymbol{L}^{p^*}(B_\varepsilon)} \leqslant C \|\boldsymbol{f}_k\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}. \tag{19}$$

Indeed, assuming, per absurdum, the invalidity of (19). Then for any $m \in \mathbb{N}^*$ there exists $\ell_m \in \mathbb{N}, \boldsymbol{f}_{\ell_m} \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{-1,2}(\mathbb{R}^3)$ and $\boldsymbol{v}_m \in \boldsymbol{\mathcal{D}}_\sigma(\mathbb{R}^3)$ such that, if $(\boldsymbol{u}_{\ell_m}, \pi_{\ell_m}) \in (\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,2}(\mathbb{R}^3)) \times (L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3))$ denotes the corresponding solution to the following problem :

$$-\Delta\, \boldsymbol{u}_{\ell_m} + \operatorname{div}(\boldsymbol{v}_m \otimes \boldsymbol{u}_{\ell_m}) + \nabla\, \pi_{\ell_m} = \boldsymbol{f}_{\ell_m}, \quad \operatorname{div} \boldsymbol{u}_{\ell_m} = 0 \quad \text{in} \quad \mathbb{R}^3, \tag{20}$$

the inequality

$$\|\boldsymbol{u}_{\ell_m}\|_{\boldsymbol{L}^{p^*}(B_\epsilon)} > m \|\boldsymbol{f}_{\ell_m}\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}, \tag{21}$$

would hold. Note that $\boldsymbol{f}_{\ell_m} = \operatorname{div}(\rho_t * \mathbf{F}_{\ell_m})$ with $\mathbf{F}_{\ell_m} = \varphi_{\ell_m}\mathbf{F}$. Set $\boldsymbol{w}_m = \dfrac{\boldsymbol{u}_{\ell_m}}{\|\boldsymbol{u}_{\ell_m}\|_{\boldsymbol{L}^{p^*}(B_\varepsilon)}}$, $\theta_m = \dfrac{\pi_{\ell_m}}{\|\boldsymbol{u}_{\ell_m}\|_{\boldsymbol{L}^{p^*}(B_\varepsilon)}}$ and $\mathbf{R}_m = \dfrac{\boldsymbol{f}_{\ell_m}}{\|\boldsymbol{u}_{\ell_m}\|_{\boldsymbol{L}^{p^*}(B_\varepsilon)}}$. Then for any $m \in \mathbb{N}^*$ we have

$$-\Delta\, \boldsymbol{w}_m + \operatorname{div}(\boldsymbol{v}_m \otimes \boldsymbol{w}_m) + \nabla\, \theta_m = \mathbf{R}_m \quad \text{and} \quad \operatorname{div} \boldsymbol{w}_m = 0 \quad \text{in} \quad \mathbb{R}^3. \tag{22}$$

Now, using (22) and the fact that $\operatorname{div}(\boldsymbol{v}_m \otimes \boldsymbol{w}_m) = \boldsymbol{v}_m \cdot \nabla\, \boldsymbol{w}_m$, we obtain for any $m \in \mathbb{N}^*$ and $t > 0$

$$\int_{\mathbb{R}^3} |\nabla\, \boldsymbol{w}_m|^2 \, dx = -\frac{1}{\|\boldsymbol{u}_{\ell_m}\|_{\boldsymbol{L}^{p^*}(B_\varepsilon)}} \int_{\mathbb{R}^3} \rho_t * \mathbf{F}_{\ell_m} : \nabla\, \boldsymbol{w}_m \, dx.$$

Using (21) and Cauchy Schwartz inequality, we have

$$\|\nabla \boldsymbol{w}_m\|_{\boldsymbol{L}^2(\mathbb{R}^3)} < \frac{1}{m\|\boldsymbol{f}_{\ell_m}\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}} \|\rho_t * \mathbf{F}_{\ell_m}\|_{\boldsymbol{L}^2(\mathbb{R}^3)}. \tag{23}$$

Using (12) and choosing $t = \dfrac{1}{m^\alpha}$ with $0 < \alpha < \frac{q'}{3}$, we deduce that

$$\|\nabla \boldsymbol{w}_m\|_{\boldsymbol{L}^2(\mathbb{R}^3)} \leqslant \frac{4\pi}{3m^{1-\frac{3\alpha}{q'}}\|\boldsymbol{f}_{\ell_m}\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}} \|\mathbf{F}_{\ell_m}\|_{\boldsymbol{L}^p(\mathbb{R}^3)}. \tag{24}$$

Because the semi-norm $\|\nabla \cdot\|_{\boldsymbol{L}^2(\mathbb{R}^3)}$ is equivalent to the full norm $\|\cdot\|_{\boldsymbol{W}_0^{1,2}(\mathbb{R}^3)}$ and the right hand side of the last inequality tends to zero when m goes to ∞, we deduce that

$$\boldsymbol{w}_m \to \boldsymbol{0} \text{ in } \quad \boldsymbol{W}_0^{1,2}(\mathbb{R}^3). \tag{25}$$

Then, $\boldsymbol{w}_m \to \boldsymbol{0}$ in $\boldsymbol{L}^6(\mathbb{R}^3)$ and in particular in $\boldsymbol{L}^{p*}(B_\varepsilon)$. On the other hand, we have $\|\boldsymbol{w}_m\|_{\boldsymbol{L}^{p*}(B_\varepsilon)} = 1$, leading to a contradiction. Inequality (19) is therefore established. From (18) , (19) and (9) we obtain for any $k \in \mathbb{N}^*$

$$\|\boldsymbol{u}_k\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + \|\pi_k\|_{L^p(\mathbb{R}^3)} \leqslant 2C_1(1 + C\|\boldsymbol{v}\|_{\boldsymbol{L}^3(\mathbb{R}^3)})\|\boldsymbol{f}_k\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}. \tag{26}$$

Thus we can extract a subsequences of $\boldsymbol{u}_k$ and π_k, still denoted by $\boldsymbol{u}_k$ and π_k, such that

$$\boldsymbol{u}_k \rightharpoonup \boldsymbol{u} \quad \text{in} \quad \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \quad \text{and} \quad \pi_k \rightharpoonup \pi \quad \text{in} \quad L^p(\mathbb{R}^3),$$

where $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ verifies (7) and the following estimate

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + \|\pi\|_{L^p(\mathbb{R}^3)} \leqslant 2C_1(1 + C\|\boldsymbol{v}\|_{\boldsymbol{L}^3(\mathbb{R}^3)})\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}. \tag{27}$$

Step 2. We suppose that $\boldsymbol{v}$ belongs only to $\boldsymbol{L}^3_\sigma(\mathbb{R}^3)$. Let $\boldsymbol{v}_\lambda \in \boldsymbol{\mathcal{D}}_\sigma(\mathbb{R}^3)$ such that

$$\boldsymbol{v}_\lambda \longrightarrow \boldsymbol{v} \qquad in \ \boldsymbol{L}^3(\mathbb{R}^3). \tag{28}$$

Using the first step, there exists $(\boldsymbol{u}_\lambda, \pi_\lambda) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ satisfying

$$-\Delta \boldsymbol{u}_\lambda + \operatorname{div}(\boldsymbol{v}_\lambda \otimes \boldsymbol{u}_\lambda) + \nabla \pi_\lambda = \boldsymbol{f} \quad \text{and} \quad \operatorname{div} \boldsymbol{u}_\lambda = 0 \quad \text{in} \quad \mathbb{R}^3, \tag{29}$$

and satisfying the estimate

$$\|\boldsymbol{u}_\lambda\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + \|\pi_\lambda\|_{L^p(\mathbb{R}^3)} \leqslant 2C_1(1 + C\|\boldsymbol{v}_\lambda\|_{\boldsymbol{L}^3(\mathbb{R}^3)})\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}. \tag{30}$$

We can finally extract a subsequence which converges to $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ which is a solution of the Oseen problem (7) and verifying the estimate (8) when $1 < p < 2$. For $p = 2$, estimate (8) was proved in Theorem 3.4 of [2]. □

We will prove now some regularity results, when the external forces belong to the intersection of negative weighted Sobolev spaces. The proof of the first result is similar to that of Theorem 1.

Theorem 2. *Let* $1 < p < 2$*. Let* $\boldsymbol{f}$ *belonging to* $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{-1,2}(\mathbb{R}^3)$ *satisfying the compatibility condition* (4) *and let* $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$*. Then the Oseen problem* (7) *has a unique solution* $(\boldsymbol{u}, \pi) \in (\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,2}(\mathbb{R}^3)) \times (L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3))$ *such that*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + \|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,2}(\mathbb{R}^3)} + \|\pi\|_{L^p(\mathbb{R}^3)} + \|\pi\|_{L^2(\mathbb{R}^3)} \leqslant C(1 + \|\boldsymbol{v}\|_{\boldsymbol{L}^3(\mathbb{R}^3)})(\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,2}(\mathbb{R}^3)} + \|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}). \quad (31)$$

In Theorem 1, we have studied the existence of weak solution of the Oseen problem when $1 < p \leq 2$. Now the question that will be discussed: if, the solution given by is Theorem 1 unique? If it is unique, is it for all $1 < p \leq 2$? The first answer is given in the following proposition:

Proposition 3. *Let* $6/5 < p < 2$*. Let* $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)$ *satisfying the compatibility condition* (4) *and* $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$*. Then the solution* $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ *given by Theorem 1 is unique.*

Proof. Suppose that there exist two solutions $(\boldsymbol{u}_1, \pi_1)$ and $(\boldsymbol{u}_2, \pi_2)$ belong to $\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ and verifying Problem (7). Set $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ and $\pi = \pi_1 - \pi_2$ then we have

$$-\Delta \boldsymbol{u} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{u}) + \nabla \pi = \boldsymbol{0} \quad \text{and} \quad \operatorname{div} \boldsymbol{u} = 0 \quad \text{in} \quad \mathbb{R}^3. \quad (32)$$

Our aim is to prove that $(\boldsymbol{u}, \pi) = (\boldsymbol{0}, 0)$. Observe that for any $\varepsilon > 0$, $\boldsymbol{v}$ can be decomposed as: $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2$ with

$$\boldsymbol{v}_1 \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3), \quad \|\boldsymbol{v}_1\|_{\boldsymbol{L}^3(\mathbb{R}^3)} < \varepsilon \qquad \text{and} \quad \boldsymbol{v}_2 \in \boldsymbol{\mathcal{D}}_\sigma(\mathbb{R}^3). \quad (33)$$

The parameter ε will be fixed at the end of the proof.

Note that $\boldsymbol{v}_2 \in \boldsymbol{L}^1(\mathbb{R}^3) \cap \boldsymbol{L}^\infty(\mathbb{R}^3)$. Now, since $\boldsymbol{u} \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \hookrightarrow \boldsymbol{L}^{p*}(\mathbb{R}^3)$ we prove that $\boldsymbol{v}_2 \otimes \boldsymbol{u}$ belongs to $\boldsymbol{L}^{p*}(\mathbb{R}^3) \cap \boldsymbol{L}^1(\mathbb{R}^3)$. As $6/5 < p < 2$, then $2 < p * 6$ and thus $\operatorname{div}(\boldsymbol{v}_2 \otimes \boldsymbol{u}) = \boldsymbol{v}_2 \cdot \nabla \boldsymbol{u}$ belongs to $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{-1,2}(\mathbb{R}^3)$ and satisfies the compatibilty condition (4). Then it follows from Theorem 2 that there exists a unique $\boldsymbol{z} \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,2}(\mathbb{R}^3)$ and $\theta \in \boldsymbol{L}^p(\mathbb{R}^3) \cap \boldsymbol{L}^2(\mathbb{R}^3)$ such that

$$-\Delta \boldsymbol{z} + \operatorname{div}(\boldsymbol{v}_1 \otimes \boldsymbol{z}) + \nabla \theta = -\boldsymbol{v}_2 \cdot \nabla \boldsymbol{u} \quad \text{and} \quad \operatorname{div} \boldsymbol{z} = 0 \quad \text{in} \quad \mathbb{R}^3. \quad (34)$$

Because of (32) and (34), the functions $\boldsymbol{w} = \boldsymbol{z} - \boldsymbol{u}$ and $q = \theta - \pi$ satisfy:

$$-\Delta \boldsymbol{w} + \operatorname{div}(\boldsymbol{v}_1 \otimes \boldsymbol{w}) + \nabla q = \boldsymbol{0} \quad \text{and} \quad \operatorname{div} \boldsymbol{w} = 0 \quad \text{in} \quad \mathbb{R}^3. \quad (35)$$

From the Stokes theory see ([1]) and Sobolev imbeddings, we obtain

$$\begin{aligned} \|\boldsymbol{w}\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} \leq C\|\boldsymbol{v}_1 \otimes \boldsymbol{w}\|_{\boldsymbol{L}^p(\mathbb{R}^3)} &\leq C\|\boldsymbol{v}_1\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\|\boldsymbol{w}\|_{\boldsymbol{L}^{p*}(\mathbb{R}^3)} \\ &\leq CC^*\|\boldsymbol{v}_1\|_{\boldsymbol{L}^3(\mathbb{R}^3)}\|\boldsymbol{w}\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} \\ &\leq CC^*\varepsilon\|\boldsymbol{w}\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)}. \end{aligned}$$

Taking $0 < \varepsilon < 1/(CC^*)$, we conclude that $\boldsymbol{w} = \boldsymbol{0}$ and so $q = 0$. Thus $(\boldsymbol{u}, \pi)$ belongs to $\boldsymbol{W}_0^{1,2}(\mathbb{R}^3) \times \boldsymbol{L}^2(\mathbb{R}^3)$ and we can write that $\operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{u}) = \boldsymbol{v} \cdot \nabla \boldsymbol{u}$. Using (32), we deduce that

$$\langle -\Delta \boldsymbol{u} + \boldsymbol{v} \cdot \nabla \boldsymbol{u} + \nabla \pi, \boldsymbol{u} \rangle_{\boldsymbol{W}_0^{-1,2}(\mathbb{R}^3) \times \boldsymbol{W}_0^{1,2}(\mathbb{R}^3)} = \boldsymbol{0},$$

and so

$$\|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2(\mathbb{R}^3)} + \int_{\mathbb{R}^3} \boldsymbol{v} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{u}\, dx = 0.$$

Since $\int_{\mathbb{R}^3} \boldsymbol{v} \cdot \nabla \boldsymbol{u} \cdot \boldsymbol{u}\, dx = \frac{1}{2} \int_{\mathbb{R}^3} \boldsymbol{v} \cdot \nabla |\boldsymbol{u}^2|\, dx = 0$, we prove that $\|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2(\mathbb{R}^3)} = 0$ and thus $\boldsymbol{u} = \boldsymbol{0}$ and so $\pi = 0$. Finally, we have proved that $(\boldsymbol{u}, \pi) = (\boldsymbol{0}, 0)$ for any $6/5 < p < 2$. □

The second regularity result is announced in the following theorem.

Theorem 4. *Let $1 < p < r < 2$. Suppose that $\boldsymbol{f}$ belongs to $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{-1,r}(\mathbb{R}^3)$ satisfying the compatibility condition (4) with respect to p and r and let $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$. Then the Oseen problem* (7) *has a solution $(\boldsymbol{u}, \pi) \in (\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,r}(\mathbb{R}^3)) \times (L^p(\mathbb{R}^3) \cap L^r(\mathbb{R}^3))$ such that*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + \|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,r}(\mathbb{R}^3)} + \|\pi\|_{L^p(\mathbb{R}^3)} + \|\pi\|_{L^r(\mathbb{R}^3)} \leqslant C(1 + \|\boldsymbol{v}\|_{\boldsymbol{L}^3(\mathbb{R}^3)})(\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + \|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,r}(\mathbb{R}^3)}). \quad (36)$$

Proof. Let $\boldsymbol{f}$ belongs to $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{-1,r}(\mathbb{R}^3)$ and satisfying the compatibility condition (4) with respect to p and with r. Then $\boldsymbol{f}$ can be written as $\boldsymbol{f} = \mathrm{div}\boldsymbol{F}$ with $\boldsymbol{F} \in \boldsymbol{L}^p(\mathbb{R}^3) \cap \boldsymbol{L}^r(\mathbb{R}^3)$. Take the same sequence $\boldsymbol{f}_k$, as in the proof of Theorem 1, which now converges to $\boldsymbol{f}$ in $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{-1,r}(\mathbb{R}^3)$.

Step 1. We suppose that $\boldsymbol{v} \in \boldsymbol{\mathcal{D}}_\sigma(\mathbb{R}^3)$. Proceeding as in the first step of Theorem 1, there exists a unique solution

$$\boldsymbol{u}_k \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,2}(\mathbb{R}^3), \quad \pi_k \in L^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$$

such that

$$-\Delta \boldsymbol{u}_k + \mathrm{div}(\boldsymbol{v} \otimes \boldsymbol{u}_k) + \nabla \pi_k = \boldsymbol{f}_k, \quad \mathrm{div}\, \boldsymbol{u}_k = 0 \quad \text{in} \quad \mathbb{R}^3 \quad (37)$$

and satisfying the estimate

$$\|\boldsymbol{u}_k\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + \|\pi_k\|_{L^p(\mathbb{R}^3)} \leqslant C_p(1 + \|\boldsymbol{v}\|_{\boldsymbol{L}^3(\mathbb{R}^3)})\|\boldsymbol{f}_k\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}, \quad (38)$$

where C_p doesn't depend on k. On the other hand, using an interpolation argument, we have also $\boldsymbol{u}_k \in \boldsymbol{W}_0^{1,r}(\mathbb{R}^3)$, because $p < r < 2$. Now proceeding as in Theorem 1, we prove that

$$\|\boldsymbol{u}_k\|_{\boldsymbol{W}_0^{1,r}(\mathbb{R}^3)} + \|\pi_k\|_{L^r(\mathbb{R}^3)} \leqslant C_r(1 + \|\boldsymbol{v}\|_{\boldsymbol{L}^3(\mathbb{R}^3)})\|\boldsymbol{f}_k\|_{\boldsymbol{W}_0^{-1,r}(\mathbb{R}^3)}, \quad (39)$$

where C_r doesn't depend on k.

Finally, $(\boldsymbol{u}_k, \pi_k)$ is bounded in $(\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,r}(\mathbb{R}^3)) \times (L^p(\mathbb{R}^3) \cap L^r(\mathbb{R}^3))$ and we can extract a subsequence denoted again by $(\boldsymbol{u}_k, \pi_k)$ and satisfying

$$\boldsymbol{u}_k \rightharpoonup \boldsymbol{u} \quad \text{in} \quad \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,r}(\mathbb{R}^3) \quad \text{and} \quad \pi_k \rightharpoonup \pi \quad \text{in} \quad L^p(\mathbb{R}^3) \cap L^r(\mathbb{R}^3). \quad (40)$$

We then verify that $(\boldsymbol{u}, \pi)$ is a solution of (7) and we have the estimate (36).

Step 2. We suppose that $\boldsymbol{v}$ belongs only to $\boldsymbol{L}^3_\sigma(\mathbb{R}^3)$. The proof is exactly the same as in Theorem 2 where we take the exponent r instead of the exponent 2. □

Now, we study the uniqueness of generalized solution when $1 < p \leqslant 6/5$:

Proposition 5. *Let $1 < p \leqslant 6/5$. Let $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)$ satisfying the compatibility condition (4) and $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$. Then the solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ given by Theorem 1 is unique.*

Proof. We proceed as in Proposition 3. Let $(\boldsymbol{u}, \pi)$ belongs to $\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ and satisfying (32). We know that $\boldsymbol{v}_2 \otimes \boldsymbol{u}$ belongs to $\boldsymbol{L}^{p*}(\mathbb{R}^3) \cap \boldsymbol{L}^p(\mathbb{R}^3)$, with $3/2 < p* \leq 2$ and thus $\operatorname{div}(\boldsymbol{v}_2 \otimes \boldsymbol{u})$ belongs to $\boldsymbol{W}_0^{-1,p*}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)$. Moreover $\operatorname{div}(\boldsymbol{v}_2 \otimes \boldsymbol{u})$ satisfies the compatibility condition (4). Using Theorem 4, we deduce that there exists $(\boldsymbol{\xi}, \varphi) \in (\boldsymbol{W}_0^{1,p*}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,p}(\mathbb{R}^3)) \times (L^{p*}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3))$ such that

$$-\Delta \boldsymbol{\xi} + \operatorname{div}(\boldsymbol{v}_1 \otimes \boldsymbol{\xi}) + \nabla \varphi = -\operatorname{div}(\boldsymbol{v}_2 \otimes \boldsymbol{u}) \quad \text{and} \quad \operatorname{div} \boldsymbol{\xi} = 0 \quad \text{in} \quad \mathbb{R}^3. \tag{41}$$

Set $\boldsymbol{\lambda} = \boldsymbol{\xi} - \boldsymbol{u}$ and $\psi = \varphi - \pi$, we have

$$-\Delta \boldsymbol{\lambda} + \operatorname{div}(\boldsymbol{v}_1 \otimes \boldsymbol{\lambda}) + \nabla \psi = \boldsymbol{0} \quad \text{and} \quad \operatorname{div} \boldsymbol{\lambda} = 0 \quad \text{in} \quad \mathbb{R}^3.$$

As in Proposition 3, we prove that $(\boldsymbol{\lambda}, \psi) = (\boldsymbol{0}, 0)$. Then we deduce that $(\boldsymbol{u}, \pi)$ belongs to $\boldsymbol{W}_0^{1,p*}(\mathbb{R}^3) \times L^{p*}(\mathbb{R}^3)$. Using again Proposition 3, we prove that $(\boldsymbol{u}, \pi) = (\boldsymbol{0}, 0)$. □

We can now summarize our existence, uniqueness and regularity results as below.

Theorem 6. *Assume that $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$.*
i) Let $1 < p \leq 2$, $h \in L^p(\mathbb{R}^3)$ and $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)$ satisfying the compatibility condition (4). Then the Oseen problem (1) *has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ such that*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + \|\pi\|_{L^p(\mathbb{R}^3)} \leqslant C(1 + \|\boldsymbol{v}\|_{\boldsymbol{L}^3(\mathbb{R}^3)}) \Big(\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + (1 + \|\boldsymbol{v}\|_{\boldsymbol{L}^3(\mathbb{R}^3)}) \|h\|_{L^p(\mathbb{R}^3)} \Big). \tag{42}$$

ii) Let $1 < p < r \leqslant 2$. Suppose that $\boldsymbol{f}$ belongs to $\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{-1,r}(\mathbb{R}^3)$ and satisfying the compatibility condition (4) with respect to p and r. Then the Oseen problem (7) *has a unique solution $(\boldsymbol{u}, \pi) \in (\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \cap \boldsymbol{W}_0^{1,r}(\mathbb{R}^3)) \times (L^p(\mathbb{R}^3) \cap L^r(\mathbb{R}^3))$ such that*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)} + \|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,r}(\mathbb{R}^3)} + \|\pi\|_{L^p(\mathbb{R}^3)} + \|\pi\|_{L^r(\mathbb{R}^3)} \leqslant C(1 + \|\boldsymbol{v}\|_{\boldsymbol{L}^3(\mathbb{R}^3)})(\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + \|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,r}(\mathbb{R}^3)}). \tag{43}$$

Finally the following existence result can be stated via a dual argument.

Theorem 7. *For $p > 2$, let $\boldsymbol{f} \in \boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)$, $h \in L^p(\mathbb{R}^3)$ and $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$. Then, the Oseen problem* (1) *has a unique solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ if $p < 3$ and if $p \geq 3$, $\boldsymbol{u}$ is unique up to an additive constant vector. In addition, we have*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)/\mathcal{P}_{[1-3/p]}} + \|\pi\|_{L^p(\mathbb{R}^3)} \leqslant C(1 + \|\boldsymbol{v}\|_{\boldsymbol{L}^3(\mathbb{R}^3)})^2 \Big(\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + \|h\|_{L^p(\mathbb{R}^3)} \Big). \tag{44}$$

Proof. On one hand, Green formula yields, for all $\boldsymbol{w} \in \boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)$ and $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$

$$\langle -\Delta \boldsymbol{u} + \boldsymbol{v} \cdot \nabla \boldsymbol{u} + \nabla \pi, \boldsymbol{w} \rangle_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3) \times \boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)} =$$

$$\langle \boldsymbol{u}, -\Delta \boldsymbol{w} - \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{w}) \rangle_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times \boldsymbol{W}_0^{-1,p'}(\mathbb{R}^3)} - \langle \pi, \operatorname{div} \boldsymbol{w} \rangle_{L^p(\mathbb{R}^3) \times L^{p'}(\mathbb{R}^3)} .$$

Taking into account that if $p > 2$, we have $\boldsymbol{w} \in \boldsymbol{W}_0^{1,p'}(\mathbb{R}^3) \hookrightarrow \boldsymbol{L}^{3p/(2p-3)}(\mathbb{R}^3)$ and since $\boldsymbol{v} \in \boldsymbol{L}^3_\sigma(\mathbb{R}^3)$ we can conclude that $\boldsymbol{v} \otimes \boldsymbol{w} \in \boldsymbol{L}^{p'}(\mathbb{R}^3)$ and consequently $\operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{w}) \in \boldsymbol{W}_0^{-1,p'}(\mathbb{R}^3)$. On the other hand, for all $\eta \in L^{p'}(\mathbb{R}^3)$,

$$\langle \boldsymbol{u}, \nabla\,\eta\rangle_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)\times\boldsymbol{W}_0^{-1,p'}(\mathbb{R}^3)} = -\,\langle \operatorname{div}\boldsymbol{u}, \eta\rangle_{L^p(\mathbb{R}^3)\times L^{p'}(\mathbb{R}^3)}\,.$$

Then problem (1) has the following equivalent variational formulation:
Find $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ such that for all $(\boldsymbol{w}, \eta) \in \boldsymbol{W}_0^{1,p'}(\mathbb{R}^3) \times L^{p'}(\mathbb{R}^3)$,

$$\begin{aligned}\langle \boldsymbol{u}, -\Delta\,\boldsymbol{w} - \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{w}) + \nabla\,\eta\rangle_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)\times\boldsymbol{W}_0^{-1,p'}(\mathbb{R}^3)} - \langle \pi, \operatorname{div}\boldsymbol{w}\rangle_{L^p(\mathbb{R}^3)\times L^{p'}(\mathbb{R}^3)} = \\ \langle \boldsymbol{f}, \boldsymbol{w}\rangle_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)\times\boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)} - \langle h, \eta\rangle_{L^p(\mathbb{R}^3)\times L^{p'}(\mathbb{R}^3)}\,.\end{aligned} \tag{45}$$

According to Theorem 6, for each $(\boldsymbol{f}', h') \in \boldsymbol{W}_0^{-1,p'}(\mathbb{R}^3) \times L^{p'}(\mathbb{R}^3)$ satisfying

$$\langle f_i', 1\rangle_{W_0^{-1,p'}(\mathbb{R}^3)\times W_0^{1,p}(\mathbb{R}^3)} = 0 \quad \text{if} \quad p' \leqslant \frac{3}{2},$$

there exists a unique solution $(\boldsymbol{w}, \eta) \in \boldsymbol{W}_0^{1,p'}(\mathbb{R}^3) \times L^{p'}(\mathbb{R}^3)$ such that

$$-\Delta\,\boldsymbol{w} - \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{w}) + \nabla\,\eta = \boldsymbol{f}', \quad \operatorname{div}\boldsymbol{w} = h' \quad \text{in} \quad \mathbb{R}^3,$$

with the estimate

$$\|\boldsymbol{w}\|_{\boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)} + \|\eta\|_{L^{p'}(\mathbb{R}^3)} \leq C(1 + \|\boldsymbol{v}\|_{\boldsymbol{L}^3(\mathbb{R}^3)})\big(\|\boldsymbol{f}'\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + (1 + \|\boldsymbol{v}\|_{\boldsymbol{L}^3(\mathbb{R}^3)})\|h'\|_{L^p(\mathbb{R}^3)}\big).$$

Observe that the mapping

$$T\,:\ (\boldsymbol{f}', h') \mapsto \langle \boldsymbol{f}, \boldsymbol{w}\rangle_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)\times\boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)} - \langle h, \eta\rangle_{L^p(\mathbb{R}^3)\times L^{p'}(\mathbb{R}^3)}\,,$$

is linear and continuous with

$$\begin{aligned}|T(\boldsymbol{f}', h')| &\leq \|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)}\|\boldsymbol{w}\|_{\boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)} + \|h\|_{L^p(\mathbb{R}^3)}\|\eta\|_{L^{p'}(\mathbb{R}^3)} \\ &\leq C(1 + \|\boldsymbol{v}\|_{\boldsymbol{L}^3(\mathbb{R}^3)})^2\Big(\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + \|h\|_{L^p(\mathbb{R}^3)}\Big)\Big(\|\boldsymbol{f}'\|_{\boldsymbol{W}_0^{-1,p'}(\mathbb{R}^3)} + \|h'\|_{L^{p'}(\mathbb{R}^3)}\Big).\end{aligned}$$

Note that $\boldsymbol{f}'$ belongs to $\boldsymbol{W}_0^{-1,p'}(\mathbb{R}^3)$ and $\boldsymbol{f}' \perp \mathbb{R}^3$ if $p \geq 3$. Thus there exists of unique $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p}(\mathbb{R}^3) \times L^p(\mathbb{R}^3)$ if $2 < p < 3$, and a unique $(\boldsymbol{u}, \pi) \in \Big(\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)/\mathcal{P}_{[1-3/p]}) \times L^p(\mathbb{R}^3)\Big)$ if $p \geq 3$, such that

$$T(\boldsymbol{f}', h') = \langle \boldsymbol{u}, \boldsymbol{f}'\rangle_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)\times\boldsymbol{W}_0^{-1,p'}(\mathbb{R}^3)} - \langle \pi, h'\rangle_{L^p(\mathbb{R}^3)\times L^{p'}(\mathbb{R}^3)}\,,$$

with

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)/\mathcal{P}_{[1-3/p]}} + \|\pi\|_{L^p(\mathbb{R}^3)} \leq C(1 + \|\boldsymbol{v}\|_{\boldsymbol{L}^3(\mathbb{R}^3)})^2\Big(\|\boldsymbol{f}\|_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)} + \|h\|_{L^p(\mathbb{R}^3)}\Big).$$

By definition of T, it follows that

$$\langle \boldsymbol{f}, \boldsymbol{w}\rangle_{\boldsymbol{W}_0^{-1,p}(\mathbb{R}^3)\times\boldsymbol{W}_0^{1,p'}(\mathbb{R}^3)} - \langle h, \eta\rangle_{L^p(\mathbb{R}^3)\times L^{p'}(\mathbb{R}^3)} = \langle \boldsymbol{u}, \boldsymbol{f}'\rangle_{\boldsymbol{W}_0^{1,p}(\mathbb{R}^3)\times\boldsymbol{W}_0^{-1,p'}(\mathbb{R}^3)} - \langle \pi, h'\rangle_{L^p(\mathbb{R}^3)\times L^{p'}(\mathbb{R}^3)}\,,$$

which is the variational formulation (45). □

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